Special Needs in Mathematics Education

Lena Lindenskov (ed)

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CURSIV
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The painting on the front cover is: Les Montagne et le Fjord, by Mads Th. Haugsted.
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Introduction

By Lena Lindenskov

A growing research area

The articles in this issue of CURSIV focus on the difficulties that children, adolescents and adults may experience when they work with and learn about mathematics. Mathematical difficulties constitute a growing research area, although investigation of reading and writing difficulties has a far greater tradition and history. As research into difficulties with mathematics is therefore a new and advancing area, there are many questions that still need to be explored; and a variety of investigative approaches are adopted in the field. The diversity of potential research questions and methodologies is also evident in this collection of articles. Previous versions of all articles were presented at the 7th conference of the Nordic Research Network on Special Needs Education in Mathematics (NORSMA7) in November 2013, held at the Copenhagen campus of the Danish School of Education, Aarhus University. This network organises biennial conferences, alternating between the five Nordic countries (see list p. 13). Additional plenaries and workshop presentations of research and development projects at NORSMA7 are available at the website.

The interest in knowing more about mathematical difficulties is shared by researchers from a range of disciplines, teachers of mathematics, special education teachers, consultants at municipal and national authorities, textbook and ICT developers, and curriculum developers. Most recently, teachers of students with dyslexia have also shown interest in the research on mathematical difficulties. NORSMA is therefore a unique meeting place offering broad and open debate involving all of the professions mentioned. This advantage is illustrated by the
articles in this issue and by the conference proceedings ranging from NORSMA2 at Örebro University, Sweden in 2003 to NORSMA6 at Agder University in 2011. All NORSMA conferences have a day set aside as “Teacher’s Day”, organised with the specific aim of creating a dialogue with the host country’s practitioners, followed by a two-day research conference. All presentations undergo a peer-review process prior to the conference. In Copenhagen, there were 250 participants during Teacher’s Day and 125 participants during the research conference. Teacher’s Day organised three invited plenaries and four workshops. The research conference organised four invited plenaries and 27 paper presentations.

Interest in the field of mathematical difficulties is growing for three reasons: Firstly, because mathematics competence is not only a key competence for specialists, but for everyone in today’s (and tomorrow’s) high-technology, globalised world; secondly, because of the trend towards inclusive education, which challenges the fields of both mathematics education and special education (Lindenskov, 2006); and thirdly, because the field of brain research has chosen numerical cognition as a principal area of focus. While astonishing results are therefore constantly being achieved in this field, the question remains as to how these results can be used to improve mathematics teaching.

This issue thus presents a diversity of foci and positions regarding how to delimit the research field. It is clear throughout that lively discussions are still ongoing about how to conceptualise mathematical difficulties and dyscalculia. A broad variety of research questions is presented, ranging from new hypotheses about neurocognitive causes to narratives of students’ experiences in sociocultural theory. The issue includes data from a wide array of groups and settings: from pre-schoolers to adult learners and teacher educators, and from experimental laboratory settings to everyday classroom experiences. Several articles concentrate on how to create supportive learning environments and, to this end, a number of factors are explored: which ICT aids, tests, mathematical activities and teacher roles are valuable, for instance.

However, this issue also reveals the narrowness of the research field. Even though mathematics in compulsory schooling in all five Nordic countries includes much more than numbers and arithmetic, almost all the articles choose to focus on numbers and arithmetic as their only mathematical domain. This narrowness is partly shared with traditional research in mathematics education, and perhaps it is a narrowness that NORSMA will have outgrown in a few years’ time.
The articles

The issue starts with articles from the two invited international plenary speakers: Ann Gervasoni from Monash University in Melbourne, Australia, and Marie-Pascale Noël from the Catholic University of Louvain, Belgium.

*Ann Gervasoni*’s “Longitudinal Progress of Australian Six-Year-Old Students Who Participated in a Mathematics Intervention Program” draws on mathematics education theories about vulnerability in mathematics learning, clinical interviews, learning frameworks and growth points. Gervasoni has been designing and conducting mathematics intervention programmes for several years, building on a social constructivist view of mathematics learning. The article explores longitudinal effects by examining the understanding of whole number concepts and arithmetic among 42 students from 11 schools from first to fourth grade. The students all came from disadvantaged social backgrounds and were among the most vulnerable in mathematics. Some were vulnerable in only one domain, some in two or three domains and a few in all four measured domains. Also, combinations of domains varied. The students participated in an intervention in first grade with 30-minute lessons, 5 days a week for a total of 10-20 weeks, depending on their progress. The lessons were designed and customised for each student. Most participants showed accelerated, maintained and extended learning three years later. Some participants showed stagnated learning after the programme stopped. Gervasoni concludes that intervention programmes must be connected to classroom mathematics.

*Marie-Pascale Noël, Laurence Rousselle,* and *Alice De Visscher* draw on neurocognitive theories in the article “Both Specific and General Cognitive Factors Account for Dyscalculia”. They refer to DSM (Diagnostic and Statistical Manual of Mental Disorders) IV to present Developmental Dyscalculia, DD, as a persistent disorder of numerical development and mathematics learning which is not a direct consequence of mental retardation, inadequate education or a sensory deficit. They provide an overview of what has been suggested as the origins of dyscalculia. Even seen only from neurocognitive perspectives, several hypotheses exist. Among various mathematics specific factors relating to very basic number processing, the authors mention deprivation of the innate number sense. Nevertheless, the authors argue on the basis of an analysis of DD children’s performance that DD arises from a difficulty in connecting magnitude with symbols: for dyscalculics, accessing the magnitude value of Arabic numbers or number words is difficult. Among general cognitive factors, the authors explore in detail recent experimental studies of one woman and approximately one hundred fourth graders, showing that hypersensitivity to interference seems to lead to difficulties in storing...
arithmetical facts in long-term memory. The authors conclude by presenting an outline of some ideas for the classroom in accordance with their research, including teacher awareness, support from specialised professionals and the use of games and activities that, for instance, support children’s comprehension of the cardinal value of numbers symbols.

Ingemar Karlsson underlines in his article “Special and Specific Educational Needs in School Mathematics” two fundamentally different views in the research literature on low achievers. The first is the Swedish educational concept of Special Educational Needs in Mathematics (SUM/SEM) for students with low performance in mathematics and a non-passing grade in relation to the current grade system. The second is the individual-centred concept that low achievers have a neurophysiological disability that reduces their ability to deal with numbers and figures, termed dyscalculia. With reference to Olof Magne, Karlsson states that just 1% of students with SUM experience difficulties in mathematics only. Karlsson terms these students specific SUM students. Incidentally, Olof Magne together with Olav Lunde and Arne Engström were the scholars in 2000 who decided to form the Nordic Research Network on Special Needs Education in Mathematics (NORSMA) (Engström, 2007). Karlsson’s own research on the performance of eighth and ninth graders in three municipalities in Scania confirms Magne’s results, as only 1% - 1.6% are specific SUM students, failing only in mathematics. Further on, Karlsson’s interviews with ten ninth graders on their own perceptions on why they fail mostly point to sociological theories on social networks as a background for difficulties in mathematics.

The four subsequent articles by Kristinsdóttir, Reimarsdóttir & Guðjónsdóttir, Schmidt, Roos and Bagger consider the school setting as a social phenomenon in which mathematical difficulties are present.

In the article “‘There Is Always Something New After Nine’ – Action Research as a Mode for Teachers to Develop a Tool for Analysing Their Pupils’ Numeracy Skills”, Jónína Vala Kristinsdóttir, Dórópea Reimarsdóttir and Hafdis Guðjónsdóttir describe from an action research perspective how a special education teacher in Island participated in developing a tool for analysing her students’ numeracy skills, and how she informed and motivated colleagues and parents.

The researchers supported the special education teacher by documenting her experiences and communicating them as narratives. In addition, they gave her an extended introduction to theories and methods from Cognitively Guided Instruction, Piagetian and Vygotskian traditions and interview guidelines in Mathematics Recovery. The teacher’s interviews with first graders provided her with new and useful perspectives on understanding students who have difficulties in mathematics. One example is condensed in the article title, citing a student...
who struggled with generalising number structures below 100 to numbers above 100. The student’s governing principle of there always being “something new after nine” could not be applied so that 110 could follow 109. Other first grade teachers expressed an interest in using the interview guidelines. Parents were also supported and encouraged to play games involving counting and to take advantage of every opportunity in daily life to exercise their children’s counting skills. The study also confirms that the work parents do with their children has a positive effect.

The article by Maria Christina Secher Schmidt “Mathematics Difficulties and Classroom Leadership – A Case Study of Teaching Strategies and Student Participation in Inclusive Classrooms” explores four second and third grade classes at two schools in Denmark. The author draws on interviews with 4 teachers and 12 students, observations of 35 mathematics lessons and essays written by 83 students. The observations in each classroom focused on two students who experienced difficulties with mathematics and one comparison student who did not. The study shows that the teachers practise dimensions of inclusive classroom leadership that are known to be successful in teaching mathematics for all students. It seems challenging for the teachers to perceive how students in difficulties with mathematics give meaning to the tasks, concepts and operations involved. The students seem to engage in the tasks like their peers. Nevertheless, in-depth analyses drawing on theoretical perspectives from Goffman, Tronvoll, Lindenskov and Brady suggest socially and institutionally oriented explanations as to why students in difficulties are not seen and supported sufficiently. The author concludes with a suggestion of working towards pedagogical norms that portray learning difficulties as normal and valuable to all children, and of working towards a classroom culture in which it is legitimate to take risks.

The institutional perspective is further developed by Helena Roos in “Inclusion in Mathematics – The Impact of the Principal”. Roos does not see students in special education needs in mathematics as having a constant internal deficiency. She sees them in a circumstance they can get in and out of, and the study is therefore grounded in a social perspective on learning in which explorations on an institutional level become important. Roos’ mention of investment from government illustrates the Swedish authorities’ view of mathematics as necessary and relevant for all children. Roos draws upon Asp-Onsjö’s theories on inclusion which state that inclusion means to be included in the mathematical practice of the classroom. Roos also draws upon Wenger’s theories of participation in her identification of five visible communities of mathematical practice at a Swedish primary school for six- to twelve-year-old students. The author interviewed three mathematics teachers, one remedial mathematics teacher and the principal at the school. Although
she found different codes of impact of the principal in the different communities, several codes recur. The most frequent is courses, but teacher competence and didactic discussions and planning also recur in the different communities. Roos concludes that there seems to be a gap between the principal’s steering and what actually happens. It seems that the principal’s impact on realised inclusion in mathematics is relatively weak.

Anette Bagger examines national tests in Sweden as both a governmental steering mechanism and a pedagogical tool in teaching and advising students in mathematical difficulties. In her article “Student Equity vs Test Equality?”, she highlights the fundamental dilemmas teachers face. Teachers are entitled and requested to provide support to students before, during and after tests, and to adjust the test if necessary. Bagger analyses test instructions for the national tests in mathematics for third graders in Sweden and interviews eight teachers in eight classes at three schools. The study is part of a large-scale longitudinal ethnographic study of the reintroduction of national tests in third grade which shows how a discourse on testing coexists with a competitive discourse and a caring discourse. Bagger presents a thorough study of student equity, which is realised when students have equal access to resources, high-quality teachers and appropriate instructional support regardless of race, class, gender and so on. Her findings suggest that national tests may contribute to the achievement variations among certain groups of students, for instance, by making the test available in Swedish and English only. Bagger concludes that, when students’ equity comes into conflict with the test’s equality, the teacher’s focus shifts from learning to controlling. Before and after the test it is easier for the teachers to engage in the discursive practice of providing support and focusing on students’ learning.

Pernille Bødtker Sunde and Pernille Pind continue the exploration of test-related issues in “Comparison of Two Test Approaches for Detecting Mathematical Difficulties”. They compare two tests which differ in several respects: being norm-referenced or criterion-referenced; having a time limit or not; and diagnosing the comprehension of most curriculum goals or only some. The data consist of test results from 59 students in second and third grade in Denmark. The authors have designed the criterion-referenced RoS/test, which focuses on arithmetic. They see constrained arithmetic comprehension by students in the early grades as a significant background for severe mathematical difficulties and low self-esteem later on. The test design is based on Snorre Ostad’s theory of task-specific strategies for solving mathematical problems. With back-up strategies, the child follows a fixed plan, such as counting fingers. With retrieval strategies, the child retrieves information that is stored on the basis of comprehension, not to be mistaken for rote learning. The test operates with two error categories: the child gives up or
the child gives an incorrect answer. According to the authors, the strength of this kind of test lies in the fact that the test provides more information about the students’ learning difficulties and ideas for further action to be taken in individually adapted interventions.

Children’s strategies seem to be influenced by working memory; as mentioned, Noël, Rousselle and De Visscher suggest that difficulties in storing arithmetical facts in long-term memory may account for mathematical difficulties. This brings studies on memory and on the possibilities for training memory to the fore. The article “Specific Training of Working Memory and Counting Skills in Kindergarten” by Kaisa Kanerva and Minna Kyttälä presents a study of 99 six-year-olds attending kindergarten in Finland. The study is the first to train different working memory components separately. The 99 children were randomly assigned to six groups, each undergoing an intervention on, respectively, working memory, visuospatial short-term memory, visuospatial working memory, verbal short-term memory, verbal working memory, and as active and passive controls. The pre- and post-tests showed an increase in the intervention students’ performance. But the increase is only associated with the re-testing effect, where re-testing in itself leads to higher performance compared to the first test. It seems that the six-year-old children were not able to utilise specific computerised working memory training in kindergarten settings. It did not improve working memory (near transfer), nor did it improve early counting skills (far transfer). As shown by the authors elsewhere, domain-specific training in mathematical skills seems to be more effective in improving early numerical performance (Kyttälä, Kanerva, & Kroesbergen, 2015).

Domain-specific training in mathematics is what Ingemar Holgersson, Wolmet Barendregt, Elisabeth Rietz, Torgny Ottosson and Berner Lindström explore in the article “Can Children Enhance Their Arithmetic Competence by Playing a Specially Designed Computer Game?” The computer game FINGU is designed for research and for practice in school and elsewhere. The game is inspired by Gibsonian theory, which is non-dualistic and non-representational, as it rejects the idea that perception and cognition are about constructing representations of the world “outside” the individual. Instead, discovering and picking up information as invariant features of the environment are central to learning. FINGU is based on the assumption that understanding mathematics entails mastery of a rich network of factual, relational and strategic knowledge. Part-whole relations of the numbers from one to ten are included in the FINGU game. FINGU was tried out in an experimental setting with 82 students aged five, six and seven in Sweden. The students played the game as part of their ordinary pre-school or school activities. Results from the pre- and post-tests were promising as the changes in four
arithmetic measures were significant. However, the delayed post-test showed non-significant results in almost all of the tests, with low or no effect sizes. It was found that the individual differences in how children used the game were large.

With Lisser Rye Ejersbo’s “Number Sense as the Bridge to Number Understanding”, we come full circle. Ejersbo combines the two fundamental viewpoints on mathematical difficulties from neurocognition and from mathematics education. Interdisciplinary research is often called for; it is more seldom actually practised. Ejersbo writes about an obstacle faced in interdisciplinary communication and practice: terminology may be superficially alike in different disciplines, but the conceptual meaning may differ. This is the case for the term number sense. For a biological/cognitive researcher like Dehaene, number sense is an innate, intuitive understanding of numbers and their magnitude and relations. For researchers in mathematics education, meanwhile, number sense is not seen as innate, but as a broad understanding and array of skills concerning numbers and operations. Ejersbo suggests as a terminological solution that the term number sense be used for the former and number understanding for the latter. Additionally, in her research, Ejersbo explores synergetic processes between the two conceptual frames. The article presents an experiment conducted in Denmark both in a kindergarten class and in a second grade class with students who had difficulties in number reading and symbolic arithmetic. It seemed that children with no formal arithmetic instruction were able to perform symbolic addition, subtraction and comparison with large two-digit numbers when they were asked to estimate the results in several tasks.

Ejersbo’s article, like the rest of the issue, is a testimony to the relevance of creating meeting places to allow different viewpoints to interact.

Note

1 The phrase ‘in difficulties/in special needs’ is deliberately used by some authors to indicate that learning problems can be due to external factors (in difficulties) not characteristics of the person(s) (with difficulties).

References


List of NORSMA conferences

- NORSMA 1 (2001): "En matematikk for alle i en skole for alle", University of Agder (UiA) and Sørlandet Support Centre for Special Needs Education (Statped), Kristiansand, Norway.
- NORSMA 3 (2005): “Mathematics Teaching and Inclusion”, University of Aalborg, Denmark.
- NORSMA 7 (2013): Danish School of Education, DPU, Aarhus University, Campus Emdrup in Copenhagen.

Presentation at NORSMA7 published elsewhere:


Additional presentations at NORSMA 7 available at:
http://edu.au.dk/forskning/omraader/fagdidaktik/konferencer/norsma7/
Longitudinal Progress of Australian Six-Year-Old Students Who Participated in a Mathematics Intervention Program

By Ann Gervasoni

Abstract

The longitudinal progress of 42 Grade 1 students (six-year-olds) who participated in an Extending Mathematical Understanding (EMU) Intervention Program in 2010 was analysed for the following three years and compared with the progress of peers in the following four whole number domains: 1) Counting, 2) Place Value, 3) Addition and Subtraction Strategies and 4) Multiplication and Division Strategies. The findings show that participation in an EMU Intervention Program was associated with accelerated learning for the majority of students and that this learning was mostly maintained and extended in subsequent years. By the beginning of 2011, the spread of students’ growth points in all domains was very similar for the EMU group and their peers.

Keywords: mathematics intervention, mathematics difficulties, mathematics assessment, number concepts, inclusion.
Introduction

Since the 1980s, primary education in Australia has been moving towards inclusion for students who experience disabilities. This means that classroom teachers appreciate the importance of inclusive educational practices, universal curriculum design and the need to differentiate instruction and tasks based on each student’s strengths. In 2008, the Australian Government ratified the Convention on the Rights of Persons with Disabilities (CRPD) and has since voiced its commitment to inclusive education in key documents and policies, including the National Disability Strategy (Council of Australian Governments [COAG], 2011), the Australian Curriculum (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2012) and the Australian Professional Standards for Teachers (Australian Institute for Teaching and School Leadership [AITSL], 2011). However, many Australian teachers struggle to meet the challenges of providing a high quality inclusive education for all.

In mathematics education, school communities recognise the importance of providing access to mathematics learning for all but acknowledge that some students fail to thrive in a typical classroom situation. Australian primary school teachers are responsible for teaching the entire curriculum, including mathematics. Emphasis is placed on teachers identifying any students who are struggling mathematically and adjusting the learning environment to better enable them to learn. One approach in some Australian schools has been to provide six-year-old students with short-term small group mathematics intervention programmes conducted by specialist teachers. This approach aims to boost students’ mathematics learning and confidence, while enabling the specialist teacher to learn about each student’s strengths and the type of instruction that enables them to thrive mathematically. This new knowledge can then be applied to designing classroom curricula and instruction so that they are more effective for all students.

This paper explores the longitudinal progress of 42 Grade 1 students (six-year-olds) who participated in the 10-20 week (between 50 and 100 thirty-minute lessons) EMU Intervention Program in 2010 as part of the Australian Government funded Bridging the Numeracy Gap in Low SES and Aboriginal Communities Project (BTNG) (Gervasoni, Parish, Hadden, Turkenburg, Bevan, Livesey, & Croswell, 2011). The questions framing the research reported are:

1. In which combinations of whole number domains (four whole number domains are reported in this study: 1) Counting, 2) Place Value, 3) Addition and Subtraction Strategies and 4) Multiplication and Division Strategies) were six-year-old students who participated in an EMU Intervention Program initially vulnerable?
2. What is the longitudinal progress over three years of six-year-old students who participated in an EMU Intervention Program in Grade 1?

3. How does the longitudinal progress of Grade 1 students who participated in an EMU Intervention Program compare with their peers?

**Vulnerability in Mathematics Learning**

Gervasoni and Lindenskov (2011) argue that there are two groups of students who are not reaching their mathematical potential and thus have special rights for mathematics education. The first group has long-term physical, mental, intellectual or sensory impairments which in interaction with various barriers may hinder their full and effective participation in society on an equal basis with others (United Nations, 2006). The less visible second group are students who underperform in mathematics due to their “explicit or implicit exclusion from the type of mathematics learning and teaching environment required to maximise their potential and enable them to thrive mathematically” (Gervasoni & Lindenskov, 2011, p. 308). The perspective that underpins the research reported in this paper is that students who have not yet experienced the opportunities necessary for them to construct the mathematical understandings needed to engage with the school mathematics curriculum are mathematically vulnerable and at risk of poor learning outcomes.

The term *vulnerable* is widely used in population studies (e.g., Hart, Brinkman, & Blackmore, 2003), and refers to students whose environments include risk factors that may lead to poor developmental outcomes. The challenge remains for teachers to identify students who are mathematically vulnerable and to design learning environments and mathematics instruction that enables all students’ mathematics learning to flourish.

**Using Clinical Interviews and Learning Frameworks to Identify Students Who Are Mathematically Vulnerable**

Clinical interviews are broadly used by teachers in Australia and New Zealand as a means of assessing primary students’ mathematical knowledge. This use is reinforced by the findings of three large-scale projects that informed assessment and curriculum policy formation in Victoria, New South Wales and New Zealand: Count Me In Too (Gould, 2000) in New South Wales, the Early Numeracy Research Project (Clarke et al., 2002) in Victoria and the Numeracy Development Project (Higgins, Parsons, & Hyland, 2003) in New Zealand. A common feature of each of these projects was the use of a one-to-one assessment interview and an
associated research-based framework to describe progressions in mathematics learning (Bobis et al., 2005). Teachers participating in each project concluded that the benefits of the assessment interview, although time-consuming and expensive, were considerable in terms of creating an understanding of what students know and can do, and for subsequently informing teaching. An important feature of clinical interviews is that they enable the teacher to observe students as they engage in mathematics tasks and determine the strategies they use and any misconceptions (Gervasoni & Sullivan, 2007). They also enable teachers to probe students’ mathematical understanding through thoughtful questioning (Wright, Martland, & Stafford, 2000). The insights gained through this form of assessment enlighten teachers more effectively about the particular instructional needs of each student than scores from traditional pencil and paper tests, the limitations of which are well established (e.g., Clements & Ellerton, 1995). Bobis et al. (2005) concluded that another benefit of one-to-one assessment interviews and associated frameworks was in moving the focus of professional learning in mathematics from the notion of students being able to reproduce taught procedures to an emphasis on students’ thinking. This is an important outcome now that it is broadly accepted that the traditional focus on taught procedures for calculating can negatively impact on students’ number sense (Clarke, Clarke, & Horne, 2006) and may impede students’ development of mental reasoning strategies (Narode, Board, & Davenport, 1993).

**Interview guidelines based on growth points**

The Early Numeracy Interview (ENI) (Department of Education Employment and Training, 2001), developed as part of the Early Numeracy Research Project (ENRP, Clarke et al., 2002), is one example of a clinical interview with an associated research-based framework of growth points that describe key stages of learning in nine mathematics domains. The principles underlying the construction of the growth points were to: describe the development of mathematical knowledge and understanding in the first three years of school in a form and language that was useful for teachers; reflect the findings of relevant international and local research in mathematics (e.g., Steffe, von Glasersfeld, Richards, & Cobb, 1983; Fuson, 1992; Wright, Martland, & Stafford, 2000; Gould, 2000); reflect the structure of mathematics; describe the mathematical knowledge of individuals and groups; and identify students who may be mathematically vulnerable. The processes for validating the growth points, the interview items and the comparative achievement of students are described in full in Clarke et al. (2002) and have been widely reported (e.g., Clarke, 2013; Clarke, 2001). To maximise the trustworthiness of the
assessment data for measuring growth, teachers followed a detailed script for presenting the assessment interview, recorded students’ responses on a detailed record sheet and followed strict guidelines for determining growth points (Clarke et al., 2002).

To illustrate the nature of the growth points, the growth points for the Multiplication and Division Strategies domain are presented in Figure 1 (Gervasoni, 2011). These highlight the thinking strategies used by students to solve tasks.

1. **Counting group items as ones** – in a multiple group situation, the student refers only to single items.
2. **Modelling multiplication and division** (when all objects perceived) to solve problems.
3. **Partial modelling multiplication and division** (some objects perceived) – solves problems where objects are not all modelled or perceived.
4. **Abstracting multiplication and division** (no objects perceived) – solves problems where no objects are modelled or perceived.
5. **Basic, derived and intuitive strategies for multiplication** – using strategies such as commutativity and building up from known facts.
6. **Basic, derived and intuitive strategies for division**.
7. **Extending and applying multiplication and division** in complex/practical situations.

**Figure 1. Growth points for the Multiplication and Division Strategies domain.**

Each growth point represents substantial expansion in knowledge along a path to mathematical understanding (Clarke, 2001). Growth points enable teachers to identify students’ zones of proximal development so that instruction may be precise, and to identify students who may be mathematically vulnerable in any domain. For example, Grade 1 students who have not reached Growth Point 1 and therefore cannot solve simple multiplicative tasks using at least a **counting group items as ones** strategy are identified as mathematically vulnerable in this domain. This signals the need for their learning opportunities to be enhanced and adjusted.

The whole number tasks in the Early Numeracy Interview (ENI) take about 15-25 minutes per student and are administered by the classroom teacher at the commencement of the school year. There are about 40 tasks in total, and given success with a task, the teacher continues with the next tasks in a domain (e.g., Place Value) for as long as the child is successful, and according to the interview script. During the BTNG project, the ENI was revised to respond to more recent research and to increase its suitability for students throughout primary school. The revised version was renamed the Mathematics Assessment Interview (MAI).
Investigating the Longitudinal Progress of Students Who Participated in an Intervention Programme

One aim of the BTNG project (Gervasoni, 2011) was to measure the mathematics progress of students over three years so that it could be determined whether the schools’ teaching strategies were improving students’ mathematics learning. This was the overall objective of the project. Classroom teachers used the MAI to assess all of the students at the beginning of the school year so that they could learn about each student’s whole number knowledge and associated whole number growth points for use when planning and customising curricula and instruction.

The growth points reached by all students in every class were calculated and analysed by the class teachers and specialist intervention teachers to identify any students who were mathematically vulnerable and who may benefit from an intervention programme to accelerate their mathematics learning and position them to learn more successfully in the regular classroom. The most mathematically vulnerable students in each class were selected for the EMU Intervention Program (Gervasoni, 2004), according to guidelines developed for the intervention (Gervasoni, 2015). Programme length depended on progress, with students completing the programme once they reached learning targets.

Participants

The participants in the BTNG project all belonged to socially disadvantaged communities, as classified by the Australian Government, and formed two groups. The first EMU group comprised the 42 Grade 1 students who in 2010 took part in an EMU Intervention Program and whose progress was measured for three subsequent years (2011-2013). The second group is the entire cohort of 2545 Grade 1 (six-year-old) to Grade 4 (nine-year-old) students who participated in the BTNG project (Gervasoni et al., 2010) from 2009-2011. These students were all assessed using the MAI and their associated whole number growth points in 2010 and 2011 were used to provide a comparative measure of mathematics knowledge 2010-2013 for all students in the study.

In 2010, 136 of 699 (19%) Grade 1 students participating in the BTNG project were prioritised for additional mathematics support. There were 55 boys (40%) and 81 girls. Of these 136 students, only 42 participated in an EMU Intervention Program. The schools’ financial and staffing resources were not sufficient to provide intervention programmes for all eligible students. Some schools provided additional support for other vulnerable students but not with the intensity required for the EMU Intervention Program.
The 42 students in the EMU group were from 11 schools in Victoria or Western Australia. One student was Indigenous, and 18 students (38%) were from families with Health Care Cards or who received an Educational Maintenance Allowance from the Government (these are measures of financial disadvantage in Australia). Three students (6%) had non-English speaking backgrounds, one (2%) had a disability and three (6%) a severe language deficit. Three students (6%) also participated in a literacy intervention programme, Reading Recovery (Clay, 1993). In the entire cohort of peers, 47% were male, 1% were Indigenous, and 28% from families with Health Care Cards or who received an Educational Maintenance Allowance. Three percent had non-English speaking backgrounds, 3% had a disability and 3% a severe language deficit. Seven percent had participated in the Reading Recovery literacy intervention programme. In summary, very few students had non-English speaking backgrounds or learning disabilities. This supports the finding of Butterworth and Laurillard (2010).

The Extending Mathematical Understanding (EMU) Intervention Program

The most mathematically vulnerable Grade 1 students in the study participated in the EMU Intervention Program (Gervasoni, 2004), which aimed to boost their learning and confidence. The program is based on a social constructivist (Cobb, Yackel, & Wood, 1992) view of learning. Students were prioritised for participation in the EMU Intervention Program on the basis of their assessment profiles and priority scores (Gervasoni, 2011) and additional information from classroom teachers. Groups of three Grade 1 students (six-year-olds) participated in 30-minutes lessons, 5 days per week for a total of 10-20 weeks (i.e. 50-100 lessons), depending on their progress. The lessons were designed and customised for each student due to the diverse range of knowledge and difficulties noted amongst those who are mathematically vulnerable. Gervasoni and Sullivan (2007) found that it was rare to find two students with the same difficulties. Each lesson focused on whole number learning with specific emphases on quantity or numerosity (including place value and counting knowledge), mathematical investigations, and open tasks involving the four operations with an emphasis on the development of arithmetic and reasoning strategies, reflection on learning, and a daily home task. EMU specialist teachers completed a 36-hour course (at master’s degree level) that focused on diagnosing children’s difficulties, mathematical pedagogical content knowledge, and instructional design to accelerate mathematics learning. Teachers needed to have completed the professional learning course, at least 25 hours of field-based learning, and a programme of professional reading in order to be accredited to teach the EMU Intervention Program.
Identifying Eligible Students for an EMU Intervention Program

The growth points (Clarke et al., 2002) were used to identify and prioritise students for participation in an EMU Intervention Program (Gervasoni, 2004). As an example, Figure 2 shows the growth point distributions for Grade 1 to Grade 4 students participating in the BTNG project in 2011. Grade 1 students on Growth Point 0 (GP0) were identified as vulnerable in the Multiplication and Division Strategies domain because it is anticipated that they would struggle to fully participate in classroom mathematics activities that assume that students can at least solve multiplicative problems using a count all strategy. Similarly, Grade 2 students who had not reached Growth Point 2 (modelling multiplication and division to solve problems when all items are perceived) were identified as mathematically vulnerable. Students’ overall growth point profiles and any vulnerability in the four number domains were used to prioritise children for the EMU Intervention Program, with those classified as Priority 1 being the first to be offered an intervention programme. The classroom teachers then implemented individual learning plans for mathematics for the remaining students on the priority list.

The data presented in Figure 2 highlight the diversity of students’ knowledge at every grade level and thus the complexity of classroom teaching. Such diversity was also evident in each of the other whole number domains and grade levels.
Progress of Students Who Participated in an EMU Intervention Program

This paper explores the longitudinal progress of six-year-old students who participated in an EMU Intervention Program in 2010. Of interest is whether the EMU Intervention Program accelerated their learning, and how their learning progressed over three years. These students were all the most mathematically vulnerable students in their class based on their MAI assessment and growth point profiles (Gervasoni, 2004). Table 1 shows the number and combinations of whole number domains for which these students were vulnerable.

<table>
<thead>
<tr>
<th>Number of domains</th>
<th>Vulnerable in Counting</th>
<th>Vulnerable in Place Value</th>
<th>Vulnerable in Add &amp; Subtraction</th>
<th>Vulnerable in Mult &amp; Division</th>
<th>No. of students vulnerable</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
<td>2 (5%)</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
<td>9 (21%)</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td>13 (31%)</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>14 (33%)</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>4 (10%)</td>
<td>4</td>
</tr>
</tbody>
</table>

| No. of students | 29 (69%) | 17 (41%) | 17 (41%) | 30 (71%) | 42 (100%) |

Table 1. Combinations of whole number domains in which students participating in the EMU Intervention Program were vulnerable in 2010.

The data suggest that although these students were all mathematically vulnerable, they were a diverse group. Some were vulnerable in only one domain (21%), some in two (31%) or three domains (33%), but only four students (10%) were vulnerable in all four domains. This is consistent with the findings of Clarke et al. (2002). Further, the combinations of domains for which the students were vulnerable varied. Clearly there is no one pattern to describe students who are mathematically vulnerable. This highlights the complexity of teaching and the need for teachers to be expert at assessing students’ current knowledge and in designing customised instruction that enables all students to thrive.
In order to show the longitudinal progress of students who participated in an EMU Intervention Program in 2010 (but not in subsequent years), their growth point distributions in 2010-2013 for each domain were calculated and compared with all students in the cohort. Figure 3 shows the growth point distributions for the Multiplication and Division Strategies domain.

Figure 3. Progressive (2010-2012) Multiplication and Division Strategies growth point distributions (beginning of the year) for the 2010 EMU group and comparison data for all Grade 1 to Grade 4 students.

For multiplication and division growth points, all students in the upper 50% of the Grade 2 distribution progressed one growth point by Grade 3, but the learning for most EMU students in the bottom half of the Grade 2 distribution appears to have stagnated.

Note that due to the BTNG project ending in 2011, longitudinal data for all the 2012 Grade 3 cohort and 2013 Grade 4 cohort were unavailable, so 2011 Grade 3 and Grade 4 data are used as indicative of the distributions that might be expected of the 2012 Grade 3 and 2013 Grade 4 cohorts. An asterix highlights this use of 2011 Grade 3 and Grade 4 data in Figures 3-6. Data for students in the EMU group were collected for 2012 and 2013 so that their longitudinal progress could be measured.

Figure 3 shows that the spread of growth point distribution for the 2010 EMU group was substantially different to their peers, with 71% of the EMU group on Growth Point 0 (GP0) compared with only 32% of their peers. However, the EMU group made substantial progress by 2011. Further, it is striking how similar the spread of growth points are in 2011 for both the EMU group and the all students group of peers. This finding was also apparent for the other three whole number domains (see Figures 4-6). These data suggest that one effect of the EMU Intervention Program in 2010 was an acceleration of whole number learning to the point that the EMU group’s growth point distribution at the beginning of 2011 (Grade 2) mirrored that of their peers. Nevertheless, while some EMU students progressed
two or three growth points in each domain across 2010-2011, some were still vulnerable in Grade 2 and remained on the lowest growth points (GP0 and GP1).

Although it is evident that overall most students participating in the EMU Intervention Program made good progress in each domain by the beginning of Grade 2 (2011), it is important to consider whether their progress continued or faded when they no longer had the opportunity afforded by an intervention programme (2011-2013). A comparison of the Grade 2 to Grade 4 Multiplication and Division growth point distributions (Figure 3) for the 2010 EMU group suggests that their learning was maintained, but that the rate of progress for many was reduced in the following years when they did not receive additional support. It is important to note that each growth point represents a significant milestone in a student’s development that may take 12 months to achieve, as opposed to smaller steps in learning that are noticeable every day (Clarke et al., 2002). As mentioned, Figure 3 shows that the students in the upper quartile (GP3 and GP4) of the EMU distribution at the beginning of Grade 2 (2011) progressed one additional growth point from 2012 to 2013 in the Multiplication and Division Strategies domain, but EMU students in the lowest quartile (GP0 and GP1) of the distribution made less progress. Overall, the rate of progress of students in the 2010 EMU group from 2011-2013 was quite consistent with the progress of their peers, except for those in the lowest quartile.

Figure 4 shows students’ progress in the Addition and Subtraction Strategies domain. Again the EMU group made strong progress from 2010-2011 but, although their learning was maintained in subsequent years, the rate of progress reduced. Of interest is the progress of both the EMU group and their peers from Grade 2 to Grade 3 (2011-2012).
For Place Value, comparisons between the growth point distributions for Grade 2 and Grade 3 for both the EMU group and their peers suggest that if students begin Grade 2 on Growth Point 1 or 2 (one and two-digit numbers respectively), then they were likely to remain on these growth points one year later. This suggests that Grade 2 and Grade 3 classroom instruction was not sufficient for these students in Place Value.

![Place Value Growth Point Distributions – 2010-2013](image)

Figure 5. Progressive (2010-2013) Place Value growth point distributions (beginning of the year) for the 2010 EMU group and comparison data (2010-2011) for all Grade 1 to Grade 4 students.

Inspection of the 2010 EMU group’s progress in the Counting domain (Figure 6) suggests that this was accelerated across Grade 1 for all students.

![Counting Growth Point Distributions – 2010-2013](image)

Figure 6. Progressive (2010-2013) Counting growth point distributions (beginning of the year) for the 2010 EMU group and comparison data (2010-2011) for all Grade 1 to Grade 4 students.

Progress for some students stagnated across the Grade 2 and Grade 3 years, particularly if they began on Growth Point 2 (count at least 20 objects) or Growth...
Point 3 (count by ones past 109 and back from 24). In contrast, the EMU students who began on Growth Point 2 at the beginning of Grade 1 all progressed to at least Growth Point 3 one year later. This suggests that the classroom Counting curriculum and instruction for Grade 2 and Grade 3 students may not have been sufficiently focused for students who were on Growth Point 2 or Growth Point 3.

Discussion

Analysis of the 2010-2013 whole number knowledge of all students participating in the BTNG project and of the 42 students who participated in an EMU Intervention Program in 2010 highlights the notable range in these students’ growth point distributions and progress, and thus the complexity of classroom teaching. The data presented in Figures 3-6 demonstrate that there were some students on much lower growth points than their peers. For example, in contrast to the majority and despite 12 months at school, some Grade 1 students were unable to count a collection of 20 items, or count all objects to solve addition and subtraction problems, or multiplication and division problems. These students have difficulty accessing and benefiting from the Grade 1 curriculum that typically assumes this knowledge. The challenge for teachers in inclusive classrooms is to enable all students to thrive.

Forty-two Grade 1 students on the lowest growth points had the opportunity to participate in an EMU Intervention Program designed to accelerate their mathematical learning and assist them to benefit more from the classroom mathematics programme. Analysis of their whole number knowledge showed that they were a diverse group but that very few were vulnerable in all four whole number domains (Gervasoni et al., 2012). Moreover, there was no one pattern to describe these students’ whole number knowledge or learning needs. The implication is that specialist teachers and classroom teachers need to be expert at assessing each student’s current mathematical knowledge, and in designing highly responsive instruction based on this assessment data. It is also important to note that very few members of the 2010 EMU group had other learning difficulties. Only few participated in the Reading Recovery intervention programme (Clay, 1993). Likewise, only few had language backgrounds other than English and only few were assessed with severe language difficulties (Gervasoni et al., 2012). Often, it is assumed that lower achievers in mathematics have a range of factors affecting their opportunity to thrive, but this was not typical of students in this EMU group.

An important focus of this paper was reporting on the longitudinal progress of students who participated in an EMU Intervention Program in Grade 1 across the subsequent three years. Classroom teachers assessed their students at the
beginning of 2011, 2012 and 2013, and this provided a measure of their whole number learning across this period. Overall, the 42 EMU students made accelerated progress across Grade 1 and achieved significant growth by the beginning of Grade 2 (2011) that was maintained after the extended summer holidays and over the following three years. Findings (Clarke et al., 2002) suggest that mean growth for six- and seven-year-old students across one year was about one growth point in each domain, although somewhat less in Place Value. By 2011, the students participating in the EMU Intervention Program in 2010 progressed from 5 to 11 growth points overall in the four domains. There were some students who did not progress in particular domains, but most improved two growth points per domain, although rarely in Place Value.

The acceleration of learning for the EMU group is obvious when comparing the growth point spread for the EMU students when they reached Grade 2 with their Grade 2 peers. Indeed, by Grade 2 (2011) and again in Grade 3 (2012) and Grade 4 (2013), the growth point spreads of both groups were very similar compared to the marked differences observed in 2010. However, an important issue apparent in the EMU group’s Grade 2, Grade 3 and Grade 4 Growth Point distributions is that learning for some students seemed to stagnate; generally it was students in the top quartile who progressed. This finding suggests that classroom instruction for Grade 2 to Grade 4 students may not be meeting the mathematics learning needs of all students. The analyses also suggest that participation in an EMU Intervention Program in Grade 1 did not accelerate learning for all students in all domains. While one-quarter of the EMU group reached the highest growth points in the 2011, 2012 and 2013 distributions for all students, a proportion of the EMU group remained mathematically vulnerable in the subsequent years.

The EMU Intervention Program was successful in enabling most Grade 1 students to progress their whole number learning beyond that anticipated in one school year. This was true even for the students who began on the lowest growth points. It was also apparent that their progress was maintained over subsequent years, although some students’ learning stagnated. This stagnation in learning was noted also for students in the entire cohort, not only for some in the EMU group. Profitable areas for further research and development are: (a) seeking insight about why some students make less progress during an intervention programme than others, and (b) designing classroom instruction for Grade 2 to Grade 4 students that is more inclusive and maximises learning for all. It is possible that it may be beneficial for an EMU specialist teacher to advise teachers in these classes about how to refine curriculum and customise instruction. It is likely that some students may benefit from more specialised mathematics instruction beyond Grade 1, and also that classroom instruction in Grade 2 to Grade 4 may
need to be more responsive to students’ individual learning needs. It is likely that maximising mathematics learning for all students may require more attention from teachers than previously thought.

**Conclusion**

The findings presented in this paper suggest that participation in a Grade 1 EMU Intervention Program was associated with accelerated learning for the majority of students and that this learning was mostly maintained and extended in subsequent years. Following their review of another intervention program, the Maths Recovery Program, Smith, Cobb, Farran, Cordray and Munter (2013) concluded that mathematics intervention programmes must be coordinated with, rather than isolated from, the classroom mathematics programme. Clements, Sarama, Wolfe and Spitler (2013) support this view, and they claim that interventions need to be scaled up in subsequent years to be most effective for students. The longitudinal data presented in this paper highlight that some groups of students stagnated in their learning at various points once the EMU Intervention Program concluded. This suggests that specialist intervention teachers have a role to play in supporting classroom teachers to provide a more inclusive learning environment in which all students may thrive.

**Acknowledgements**

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**References**


Dansk resumé

Den langsigtede udvikling i matematikkundskaber for seks-årige elever, som deltog i en matematikintervention

I artiklen undersøges de langsigtede resultater af interventionsprogrammet ‘Extending Mathematical Understanding (EMU)’ blandt 42 elever i en australsk 1. klasse (i Australien er elever i 1. klasse seks år gamle). Interventionsprogrammet blev gennemført i 2010, og udviklingen i elevernes matematikkundskaber blev over de efterfølgende tre år løbende analyseret og sammenlignet med udviklingen blandt eleverne i en kontrolgruppe, som ikke deltog i programmet. EMU-programmet fokuserede på talforståelse og regneoperationer i forbindelse med hele tal. Resultaterne viser, at deltagelsen i EMU-programmet for størstedelen af elevernes vedkommende øgede læringen, og at denne læring i de efterfølgende tre år hovedsagelig blev bibeholdt og yderligere udbygget. På alle de undersøgte matematiske områder var fordelingen af vækstpunkter (et centralt begreb i EMU-programmet) hos eleverne i interventionsgruppen meget lig kontrolgruppen.

Nøgleord: matematisk intervention, matematikvanskeligheder, undersøgelse af elevers matematikforståelse, talbegreber, inklusion.
Both Specific and General Cognitive Factors Account for Dyscalculia

By Marie-Pascale Noël, Laurence Rousselle, & Alice De Visscher

Abstract

This article gives an overview in two parts of suggested origins of developmental dyscalculia (DD) seen from neuropsychological perspectives. The first part concerns factors that relate to very basic number processing and concludes that DD could arise from a difficulty in accessing the magnitude value of symbolic numbers (Arabic numbers or number words). The second part concerns the role of general cognitive factors. In particular, the authors present two recent studies showing that hypersensitivity to interference could lead to difficulties in storing arithmetical facts in memory. Lastly, the question is raised of how to use new neuropsychological constructs to inform teachers’ decisions in classrooms, and some ideas for adaptations in the classroom based on the described constructs are sketched.

Keywords: dyscalculia, number magnitude, symbolic numbers, arithmetic, arithmetical facts, interference, memory.
Introduction

Developmental dyscalculia (DD) is a persistent and specific disorder of numerical development and mathematics learning that is not the direct consequence of intellectual disability, inadequate education, or a sensory deficit (see DSM IV, American Psychiatric Association, 1994).

The difficulties encountered by people with DD are varied; they can concern the mastery of the symbolic number systems (permitting one to read and write Arabic numbers and to understand the base-10 system), the storage of arithmetic facts in long-term memory (for example, remembering that $5 + 4 = 9$ or $8 \times 3 = 24$), or the achievement of calculation procedures or problem solving.

Various proposals to account for these difficulties have been made. Some assume the existence of a basic number deficit; others emphasise the role of general cognitive factors. In this paper, we will develop one hypothesis for each of these proposals: basic deficit in the construction of an accurate number representation and hypersensitivity to interference.

A basic number deficit

Some authors have hypothesised that a purely numerical deficit could result in dyscalculia. For example, Butterworth (1999, 2005) suggested that DD is the consequence of a malfunction affecting the basic representation of the number magnitude. Thus, DD children would be deprived of ‘number sense’ or the representation of the quantity (Landerl, Bevan, & Butterworth, 2004). Therefore, any number processing would be problematic.

But what is this ‘number sense’? A series of studies has shown that babies are born with an innate ability to represent numerical quantity in an approximate way. This system is called the Approximate Number System or ANS (Feigenson, Dehaene, & Spelke, 2004; Lipton & Spelke, 2003, 2004; Xu, 2003; Xu & Spelke, 2000; Xu, Spelke, & Goddard, 2005). This system permits one to distinguish between the numerosity (or number of elements) of two collections that differ from each other by a sufficient ratio. At six months, the child can distinguish collections that enter into a $1/2$ ratio (such as 8 and 16, for example) but not those that enter into a $2/3$ ratio (like 8 and 12 or 16 and 24, see for example Xu & Spelke, 2000). At nine month of age, the child can distinguish collections that differ by a ratio of $2/3$ but not those within a $3/4$ ratio (Xu & Ariaga, 2007). The threshold of discrimination is called the number acuity. This acuity evolves throughout childhood (Halberda & Feigenson, 2008). Some authors consider this the basis of future numerical learning, including arithmetic and mathematics. Indeed, it has been shown that
the measurement of the ANS acuity taken at age 14 is correlated with performance in mathematics at different school ages, including kindergarten (Halberda, Mazzocco, & Feigenson, 2008).

Along similar lines, several researchers have proposed that DD could result from a malfunction of this innate representation of numerical quantity (Butterworth, 1999, 2005; Landerl, Bevan, & Butterworth, 2004; Wilson & Dehaene, 2007). In support of this hypothesis, several studies have shown that the ANS acuity of DD children was poorer than that of control children. Thus, in a task requiring the selection, without counting, of the larger of two collections of dots, DD children are less efficient than the control children when collections are numerically close (Mazzocco, Feigenson, & Halberda, 2011; Piazza et al., 2010). Similarly, Mussolin, Mejias, and Noël (2010) observed that DD children were slower and less accurate than control children when they had to compare two small but close numerosities. Finally, differences also appeared in numerical estimation tasks where the children had to estimate the number of dots that were quickly displayed. In these tasks, DD children’s estimates were more variable and less accurate than those of children without learning disabilities (Mejias, Mussolin, Rousselle, Grégoire, & Noël, 2012; see also Mazzocco, Feigenson, & Halberda, 2011).

Access to the representation of number magnitude from symbolic codes

However, other studies have provided results that do not support this hypothesis. On the one hand, in a population of unselected children, several studies failed to find any significant correlation between the ANS acuity measured in collection comparison tasks and performance in a mathematics test (e.g. Holloway & Ansari, 2009; Mundy & Gilmore, 2009, and see De Smedt, Noël, Gilmore, & Ansari, 2013 for a review). On the other hand, several studies comparing the performance of DD children and children without learning disabilities in collection comparison tasks, have failed to find any differences between the two groups (de Smedt & Gilmore, 2011; Iuculano, Tang, Hall, & Butterworth, 2008; Landerl & Kölle, 2009; Rousselle & Noël, 2007). However, in all these studies, DD children were less efficient than control children in comparison tasks of Arabic numbers. These data support the hypothesis of Rousselle and Noël (2007) that the core deficit in dyscalculia does not concern the ANS itself but rather the access to the representation of number magnitude from symbolic codes (i.e. numbers presented in Arabic digits or in words).
Table 1:

Arabic number and dot collection comparison in children with dyscalculia (DD) or in control children with no learning difficulties (C).

<table>
<thead>
<tr>
<th>References</th>
<th>Age</th>
<th>Arabic digits</th>
<th>Dot collections</th>
</tr>
</thead>
<tbody>
<tr>
<td>De Smedt &amp; Gilmore (2011)</td>
<td>-</td>
<td>DD &lt; C</td>
<td>DD = C</td>
</tr>
<tr>
<td>Rousselle &amp; Noël (2007)</td>
<td>7 years old</td>
<td>DD &lt; C</td>
<td>DD = C</td>
</tr>
<tr>
<td>Landerl, Bewan &amp; Butterworth (2004)</td>
<td>8-9 years old</td>
<td>DD &lt; C</td>
<td>-</td>
</tr>
<tr>
<td>Iuculano et al. (2008)</td>
<td>8-9 years old</td>
<td>DD &lt; C</td>
<td>DD = C</td>
</tr>
<tr>
<td>Landerl &amp; Kölle (2009)</td>
<td>8, 9, 10 years old</td>
<td>DD &lt; C</td>
<td>DD = C</td>
</tr>
<tr>
<td>Landerl, Fussenegger, Moll &amp; Willburger (2009)</td>
<td>8, 9, 10 years old</td>
<td>DD &lt; C</td>
<td>DD &lt; C</td>
</tr>
<tr>
<td>Piazza &amp; al. (2010)</td>
<td>10 years old</td>
<td>-</td>
<td>DD &lt; C</td>
</tr>
<tr>
<td>Mussolin, Mejias &amp; Noël (2010)</td>
<td>10-11 years old</td>
<td>DD &lt; C</td>
<td>DD &lt; C</td>
</tr>
<tr>
<td>Price, Holloway, Rasanen, Vesterinen, &amp; Ansari (2007)</td>
<td>12 years old</td>
<td>-</td>
<td>DD &lt; C</td>
</tr>
<tr>
<td>Mazzocco, Feigenson, &amp; Halberda (2011)</td>
<td>14 years old</td>
<td>-</td>
<td>DD &lt; C</td>
</tr>
</tbody>
</table>

How can we explain these contradictory results? If we list all the studies that compared the performance of DD and control children in tasks requiring magnitude processing of non-symbolic (set of points) or symbolic numbers (Arabic numerals or verbal numbers), a clear age effect appears (see Table 1). Indeed, with respect to non-symbolic tasks, dissociation occurs between the studies that tested young children (6-9 years) and those that tested older children (10 and over): only the latter found significant differences between the two groups at the expense of DD children. On the contrary, for the comparison of symbolic numbers (Arabic digits), DD children’s performance was systematically poorer than that of control children, whatever the age of the children tested. Thus, the first deficit observed in DD children would not affect the processing of number magnitude in itself but rather the access to number magnitude indicated by symbolic codes (Arabic numerals or words). The deficit observed in the collection process appears only secondarily. Given this observation, we must first admit that the hypothesis of a
deficit of the ANS as the basis for DD does not hold water. Indeed, if this was the case, difficulties in collections comparison should be observed at a young age. Besides, if the difficulties in processing symbolic numbers were explained by their connection to a defective ANS, they could only be secondary to the observation of a deficient ANS, and not precede it. Another interpretation is therefore required.

Some developmental theories suggest that learning a number symbol system triggers the creation of a new number representation system in the human ontogeny (see Carey, 2001, 2004; Wiese, 2003a, 2003b, 2007; Noël, Grégoire, Meert, & Seron, 2008). With a semantic content based on the ordinal information included in the sequence of symbolic numbers (counting string), this new representation would represent the exact numerical value, unlike the ANS, which is an approximate representation of the number magnitude. Indeed, when the child has to understand the cardinal value of number words, he/she has to reach a precise representation. ‘Seven’ or ‘7’ refers to the exact amount of seven, not six or eight.

**Figure 1:**

This distinction between an innate approximate numerical representation, on the one hand, and an exact representation of the number that would develop through symbolic numerical codes, on the other hand (see Figure 1), provides a new understanding of the performance profile observed in DD. Indeed, as shown in Table 1, the first difficulty observed in DD children concerns their ability to handle the magnitude representation of symbolic numbers. One might therefore consider that the basic deficit in DD children concerns the development of an accurate number representation elaborated through the learning of a symbolic number system. A difficulty in this respect is actually observed at any age and consistently across studies.

But why, later in development, do DD children show difficulty, compared to control children, in comparing collections? According to the developmental model of Carey (2009), numerical symbols lead the child to develop an accurate representation of the number, which then connects to the approximate numerical
representation of the ANS. One might therefore assume that this connection leads to a refinement of the representations of the ANS, that is to say, to an increase in its acuity. One could also imagine that the handling of exact numerical representations in exact calculation also leads to a refinement of the ANS representations. In DD children who are struggling to develop an accurate representation of the number from the symbols and manipulate this representation in arithmetic, this refinement process may be delayed or less effective. Therefore, an initial difficulty in processing the magnitude of symbolic numbers could slow down the process of refining the acuity of the ANS. This would lead to subsequent and significant differences between DD and control children in tasks measuring the acuity of the ANS. This hypothesis seems plausible. Indeed, Piazza, Pica, Izard, Spelke, and Dehaene (2013) studied the Mundurucu people, whose language contains only a few words for numerical quantities and who do not develop oral or written counting systems. Currently, some of these people have access to school and learn the Portuguese number system. The authors observed that Mundurucus who had not gone to school have an ANS acuity equivalent to the one measured in Italian six-year-old children. However, those Mundurucu people who went to school and received a mathematical education showed a better number acuity than the others. Therefore, the acuity of the ANS increases both through natural age-related maturation (as seen in babies) as well as through education and in particular, via the learning of a symbolic number system and its use in numerical tasks such as counting and calculation.

In summary, the analysis of DD children’s performance leads us to postulate the existence of a basic deficit in the construction of an accurate number representation. This difficulty would disrupt the process of refinement of the ANS and lead later on to immaturity in the development of these approximate representations, which would result in reduced numerical acuity.

Hypersensitivity to interference

Other authors were interested in the role of general cognitive factors in typical and atypical numerical development of the child. Many general cognitive factors have been considered. In the context of this article, we focus on the role that general cognitive factors can play in very specific numerical learning, namely, the acquisition of arithmetic facts. Indeed, this acquisition has been the subject of much research and is very often problematic for children with DD. Many DD children calculate answers instead of retrieving them from memory and when they try to recover them from memory, they often make mistakes (Geary, 2005). In connection with these observations, several hypotheses have been proposed.
First, it has been suggested that establishing, in long-term memory, an association between the terms of a calculation and its answer requires the co-activation in working memory of the representation of both the problem and the answer. This would be particularly difficult for people with dyscalculia for two reasons. First, when DD people solve a calculation, they often use immature counting strategies that are long and effortful so that the problem is no longer active in memory when they obtain the answer. Second, DD people could have reduced short-term memory capacity (Geary, 2005), which would not allow them to co-activate the problem and the answer together.

The second hypothesis proposed is that people with dyscalculia could have general difficulties in storing or retrieving information in long-term memory (Geary & Hoard, 2001; Geary, 2005).

Third, it has been argued that problem-response associations are often learned by rote in the form of verbal routines (Dehaene & Cohen, 1997) and accordingly, it has been hypothesised that people with dyscalculia could have a specific difficulty in storing or retrieving such verbal routines or processing phonological aspects of language (Wilson & Dehaene, 2007).

Fourth, people with dyscalculia might have inhibition difficulties and therefore may not manage the interference created by other arithmetic facts activated during the presentation of the problem.

An examination of each of these assumptions shows that low working memory capacity (central executive component, Noël, Seron, & Trovarelly, 2004; Swanson & Jerman, 2006) and low phonological awareness (De Smedt, Taylor, Archibald, & Ansari, 2009) may partly explain these learning difficulties. However, problems of information retrieval from long-term memory (Mussolin & Noël, 2008) or of inhibition (Censabella & Noël, 2008; van der Sluis, de Jong, & van der Leij, 2004) do not seem to be good candidates to explain difficulties in learning arithmetic facts.

A case study of a woman who has difficulty storing arithmetical facts in memory

Recently, De Visscher and Noël (2013) proposed a new hypothesis regarding severe difficulties in learning arithmetical facts. These authors did a comprehensive study of a person, DB, with a very high IQ and good mathematical reasoning but specific difficulties in building a network of arithmetic facts in memory. Since the primary grades, DB had found mental calculation problematic, particularly when learning the multiplication tables. A thorough cognitive assessment showed that DB performed well in tests measuring attention, executive function, and
short- and long-term memory. However, in a global mathematics test, she performed abnormally slow (-5.56 standard deviations from the mean (SD)). This slowness was also seen in a task that involved solving single-digit additions (-3.13 SD) and even more strongly when she was asked to perform single-digit multiplications (-7.43 SD).

What prevented this intelligent and determined woman from creating a network of arithmetic facts in long-term memory? De Visscher and Noël (2013) first tested different assumptions found in the literature: a deficit in the number magnitude representation, a problem in accessing the magnitude of symbolic numbers, or a problem in encoding or retrieving verbal routines from long-term memory. The results of these investigations clearly showed that DB had no impairment of these processes. Thus, another explanation had to be found.

Research on the arithmetical facts network in the typical adult shows that the different elements of the network are strongly associated with each other. The presentation of a problem evokes the correct response associated with it but also activates other responses that are associated with one or other of the calculation terms (e.g. 6 x 4 and 18) or even answers associated with the terms of the calculation, but in another operation (for example, 5 + 3 and 15, Campbell & Timm, 2000). So this is a network with much interference (Campbell, 1995; Verguts & Fias, 2005). Barrouillet, Fayol and Lathulière (1997) proposed that people with dyscalculia have difficulty managing the interference created by other arithmetic facts activated during the presentation of the problem.

On this basis, one could easily imagine that sensitivity to interference would not only cause problems when retrieving the response from memory, but would also cause difficulties when learning these facts and constituting this network. Indeed, at the time of being stored in memory, features shared by different items cause interference and affect the storage (Oberauer & Kliegl, 2006). This interference is described as proactive (Jonides & Nee, 2006). Thus, when a child learns arithmetic facts, he/she has to store associations of items that share many features, in this case digits (e.g. in ‘6 x 4 = 24’ and ‘7 x 4 = 28’, the two problems share the digits 4 and 2), which introduces interference. Consequently, one can imagine that people with high sensitivity to interference would find it more difficult to encode arithmetic facts.

This sensitivity to interference was measured in DB with several paradigms. Two of them are described here. First, we evaluated DB’s ability to memorise combinations of pairs of words: words that are very different from each other (associated pairs of Wechsler III) or close pairs of words (first names associated with family names). While DB had no difficulty with dissimilar words, her performance was clearly deficient for similar words. A second task required the storage
of associations between a first name, a last name and a country, for example, ‘Gilles Arlot lives in Benin’. The person had to store the country in which a given person was living. In some cases, the same first or last name was shared. For instance, the participant had to learn that ‘Gilles Arlot lives in Benin’ and also that another ‘Gilles’ lives in another country. In those cases, the level of interference was even higher. Again here, DB showed a hypersensitivity to interference with clearly impaired performance when she had to memorise associations that shared common features.

In summary, this single-case analysis showed that a specific difficulty in learning arithmetical facts could be due to a difficulty in learning associations of items with several overlapping features, which induce interference.

**A study on low-performing fourth-graders**

This observation of an association of disorders in a single-case study is, however, not very strong proof of this hypothesis. Following the first study, De Vissher and Noël (2014a) conducted another study to test their hypothesis on a larger sample. One hundred children from 4th grade (9-10 years old), the grade at which children in school learn to memorise the multiplication tables, were tested collectively to examine their fluency in solving small addition and multiplication calculations. From the study cohort, two groups of 23 children were constituted: children with the lowest performance (low group) and control children with normal performance (typical group). These two groups were matched for age, gender, and school classroom. An individual computerised assessment of their multiplication capacities confirmed this distribution of children. Then, these two groups of children were subjected to a memory task in which the interference level was manipulated. In this task, three associations between a character and a location (in the form of pictures: for example, the picture of Asterix next to a mountain picture) were successively presented to the child, who had to memorise them. In the testing phase the pictures of characters and locations were presented to the child, who had to verify whether the associations corresponded to the ones he/she had just learned. The first block was followed by another block of the same type, which combined the same characters and places differently. Again, the successive presentation of the three character-location pairs was followed by a test phase. Then, other characters and other places were presented and the procedure was repeated five times.

The manipulation of the interference happened in the test phase (where half the items are true). In both the true and the false items, some items interfered a great deal while there was less interference with the others. For example, if in
block 1, ‘Asterix – Mountain’ and ‘Marsupilami – the store’ were presented and in the test phase, ‘Asterix – Mountain’ was shown, then this was true and not interfering. However, if ‘Marsupilami – the mountain’ was shown, this pair did not match the original and was false and interfering since in the original showing this place and character were presented, but not together. Similarly, if in the next block, ‘Asterix – the store’ was presented and in the test phase, ‘Asterix – Mountain’ was shown, this item was false and interfering since it corresponded to a former association that interfered with the new learning.

Analyses of the results of this task show that the two groups of children are equally good at verifying the items with low interference. However, for the interfering items, a highly significant difference appears between the two groups, with higher performance for the typical group than for the low-arithmetical fact group.

These results suggest that children with a weak arithmetic fact network show greater sensitivity to memory interference than typically developing children. These results are in line with those of Passolunghi, Cornoldi, and De Liberto (1999), who showed that children with dyscalculia had difficulties in working memory tasks in which they had to inhibit the information that was no longer relevant.

Note, however, that this sensitivity to interference in memory does not correspond to an overall inhibition deficit, since these two groups of children did not differ in their degree of ability to resist a dominant response in the classical Stroop task (De Visscher & Noël, 2014b). Similarly, several studies have shown that DD children can inhibit a dominant response as in the Stroop paradigm as easily as most other children (Censabella & Noël, 2008, van der Sluis, de Jong, & van der Leij, 2004).

In summary, dyscalculia is a disorder that can take various forms and has multiple causes. We conclude that specific as well as general cognitive factors may account for dyscalculia. We have underlined one hypothesis related to specific basic number deficits: the hypothesis of a difficulty in building an accurate representation of a symbolic number’s magnitude. Dyscalculia may also be due to a more general cognitive deficit. We have underlined the hypothesis that a very high sensitivity to interference could be related to difficulty in encoding arithmetic facts in long-term memory.

**Ideas for the classroom**

The question arises whether neuropsychological evidence and theoretical constructs can be used in classrooms. We end this article by presenting some principles and ideas that in our view ought to be further examined by educational
professionals. We suggest that evaluation and rehabilitation of dyscalculia should take into account the heterogeneity of the possible forms and causes shown above. Teachers should also be informed of these specific learning disabilities and contact specialised professionals when they suspect this type of problem. Moreover, knowing the usual difficulties in the child’s numerical development, teachers could add some games to their pallet of tools.

To enhance children’s comprehension of the cardinal value of number words or Arabic digits, teachers could propose some specific exercises. Indeed, it is not sufficient to teach pre-schoolers the counting list or the counting procedure for enumerating sets of items; the emphasis should be on the cardinal value of the number words. For instance, the teacher could present a collection of items (e.g. seven apples) for a short time (e.g. two seconds) and ask the children to guess how many items there are. Then the children could count the collection and get feedback on their estimates. Conversely, they might be asked to grab seven tokens at once, and then count them to see how good their approximation was.

Also, using a number line and asking children to position numbers on that line reinforces the link between the exact representation of number symbols and the analogue representation of the spatial number line. Several intervention studies using spatial tools have resulted in significant improvements in DD children (e.g. Wilson, Revkin, Cohen, Cohen, & Dehaene, 2006). Another number line game developed by Ramani and Siegler (2008) has proved beneficial for kindergarten children. A number line made of 10 coloured squares is placed in front of the children with a number from 1 to 10 written on each of the squares. The game is a race in which children have to move a token from Square 1 to Square 10. The child uses a spinner to see how many steps (1 or 2) he/she is allowed to move. While moving his/her token, the child has to read the numbers out loud (e.g. a child on Square 3 who spins a 2 should say ‘four, five’ as he/she moves the token). For older children, other spatial games have been developed in which they have to estimate, on a number line, the approximate result of an arithmetical operation (Vilette, Mawart, & Rusinek, 2010; Kucian et al., 2011). All these exercises will help children make sense of symbolic numbers and arithmetical operations.

Much less is known about how to deal with the problem of high sensitivity to interference in learning arithmetic facts. However, acknowledging that this memorisation process may be especially difficult for some children with this particular memory problem is a good first step. Indeed, in these cases, it is useless to blame the child or ask him/her to copy multiplication tables an endless number of times. Rather, it might be more beneficial to allow him/her to have access to the Pythagorean table so that she/he can circumvent this impairment and have the possibility of tackling more complex problems.
References


Both Specific and General Cognitive Factors Account for Dyscalculia


Dansk resumé

Specifikke såvel som generelle kognitive faktorer kan forårsage dyskalkuli

Denne artikel giver en oversigt over de forslag, der i tidens løb har været til, hvad der kan være årsag til dyskalkuli, DD. Første del omhandler kognitive faktorer, der relaterer til helt basal talopfattelse. Her konkluderer artiklens forfattere, at dyskalkuli kan stemme fra en vanskelighed med at opfatte størrelsesværdien af symboliserede tal, det være sig arabertal eller talnavne på modersmål og andre sprog. Den anden del omhandler generelle kognitive faktorer. Forfatterne præsenterer to nylige studier, som peger på, at hypersensitivitet over for interferens fører til vanskeligheder med at lagre aritmetiske fakta i hukommelsen. På basis af de behandlede teoretiske begreber skitseres afslutningsvis nogle idéer til, hvordan
artiklens resultater kan anvendes til at forbedre matematikundervisningen. Disse idéer anbefales til videre overvejelser og afprøvning.

Nøgleord: dyskalkuli, talstørrelse, symboliserede tal, aritmetik, aritmetiske fakta, interferens, hukommelse.
Special and Specific Educational Needs in School Mathematics

By Ingemar Karlsson

Abstract

The background for the study is twofold: Firstly is the declining performance in mathematics of Swedish students, as measured in international studies; secondly are discussions in Swedish mathematical educational research of the phenomena and concept of dyscalculia compared to other concepts for what constitutes and what causes mathematical difficulties. The study investigates low performance in mathematics measured by ratings in three municipalities. For analysing these ratings, the concepts of special and specific educational needs in mathematics are used. The low percentage of students in specific educational needs in mathematics (1%) is discussed in relation to the percentage put forward by proponents for dyscalculia. According to Shalev (2004), the prevalence of developmental dyscalculia is 5 to 6% in the school-aged population. Additionally, as a new perspective to be followed up in research in mathematics education, the study investigates ten Grade 9 students’ own perceptions of what causes their low performance in mathematics (students in Grade 9 are approximately 15 years old). In these students’ own perceptions, causes for their mathematical difficulties are to be found in social networks, motivation and other cognitive difficulties than dyscalculia.

Keywords: dyscalculia, mathematics difficulties, special educational needs in mathematics (SUM), specific educational needs in mathematics (specific SUM), social network.
Background

In Sweden, there is currently an intense debate on improving students’ mathematical skills. International studies show that Swedish students perform below the level that could be expected when compared to students in a number of other OECD countries. This has prompted discussions on how to promote more effective learning of mathematics in Swedish schools. With compulsory schooling, it follows in Sweden that all students have the right to reach the goals that are set. If a student risks not reaching the goals that are set, he or she must quickly be offered support in the form of special adjustments, as well as interventions (Skolverket, 2015, p. 3). Against this background, it becomes a central task for Swedish schools to provide extra support and special education in order to ensure that all students achieve a passing grade. Naturally, the students who are low achievers in the subject are in particular focus. Several researchers argue that when these low achieving students leave school, they encounter both social and emotional exclusion because they lack mathematical skills; as such, low performance in mathematics is, according to some researchers, one of the major problems at school (Every Child a Chance Trust, 2009).

However, there is also a feeling that this problem does not get the attention it deserves. Clarifying terminology to describe low performance is a first step to increasing this attention. In discussions within Swedish educational research regarding the phenomena and concepts of mathematical difficulties, the concept dyscalculia has been proposed by some and refuted by others, who prefer other concepts for determining what constitutes and what causes mathematical difficulties. This study contributes to these discussions by presenting the concepts of special educational needs in school mathematics (SUM) and specific educational needs in school mathematics (specific SUM, using these concepts to critically discuss the dyscalculia concept, investigating the proportion of students who do not succeed in mathematics, and setting up a small-scale qualitative study on students’ own explanations.

Research questions

The overall aim of this empirical study was twofold. Firstly, it aimed to investigate the numbers of SUM students and of specific SUM students. Magne (2006) suggests the term SUM students, i.e. students with special needs in mathematics, for all students who fail to meet the requirements in mathematics. With the term specific SUM students, Magne refers to students with specific mathematical difficulties, that is, who fail to meet the requirements only in mathematics and not
in any other subjects. This means that Magne defines specific SUM students as a subset of SUM students.

Secondly, the empirical study aimed to investigate the students’ own explanations of their mathematical difficulties. Even among the students belonging to the specific SUM group, and thus identified as having a particular learning difficulty in relation to mathematics, there may be students who exhibit a whole range of difficulties, although they do not show up as failing in other school subjects. Interviews with students from both the group with SUM and the group with specific SUM were organised to reveal the factors that might influence student performance: school attendance, educational background and previously encountered teaching methods can all have an impact on current performance. Moreover, the extent to which the student received help to overcome any mathematical difficulties and the kind of help that was offered were investigated through the student interviews: Have the students in the two groups been treated differently with respect to in-class support? The students were also asked about when their difficulties began. In this way, the interviews sought to capture also other possible causes of the SUM students’ and the specific SUM students’ problems than those rooted in neurophysiology.

On this basis, the following research questions were drafted:

- What proportion of students in Grades 8 and 9 (students in Grade 8 and 9 are approximately 14 and 15 years old respectively) in three municipalities in Scania, southern Sweden have special educational needs in mathematics (SUM)?
- What proportion of these students can be considered to have specific SUM?
- What are the reasons given by students with low achievement levels in mathematics for their performance?

**Theoretical framework for students with low achievement in mathematics**

How do we understand the concept of low achievers in mathematics? Two different views or paradigms are presented in the research literature. The first is the concept of Special Educational Needs in Mathematics (as mentioned above, SUM is the Swedish abbreviation (Magne, 2006)). A student who has low performance in mathematics and a non-passing grade in relation to whichever grading system is in use falls into this category. The second view is that low achievers have a neurophysiological disability that reduces the ability to deal with numbers
and figures. This difficulty is called dyscalculia, and usually only affects certain aspects of mathematics (Lundberg & Sterner, 2009).

It was during the 1930s that researchers first began to suggest that difficulties in mathematics were due to specific impairments. How the difficulties of the individual student are explained has evolved to include complex neurological examinations, and the term dyscalculia is used in medicine as well as in neuropsychology as a model to explain mathematical difficulties. However, a definition of dyscalculia, as well as explanations of the phenomenon, remains elusive. These difficulties in finding a comprehensive and inclusive definition of the concept of dyscalculia have made the term a controversial one.

It is important to emphasise that in Sweden the term dyscalculia is not related to the definitions of SUM or specific SUM.

If a student has learning difficulties, then this student will usually perform below the expected level in numerous school subjects, including mathematics. Engström and Magne (2003) showed in their Middletown survey that SUM students accounted for 15% of the Grade 9 students in an average Swedish municipality, representative of the country as a whole. According to Magne (1994, 2006), the students assessed as having specific SUM are statistically rare and usually represent around 1% in Sweden. This empirical study was therefore based on the working assumption that the number of students with specific SUM may be lower than 5 to 6% (Shalev, 2004).

In the Swedish Science Council report series relating to special education, a number of discussions have relevance to the area of mathematics (Rosenqvist, 2007). Studies have found that, when discussing how best to combat learning difficulties, authorities, school administrators, school staff and the general public often base their ideas less on scientific evidence regarding how to aid those with special educational needs, and more on what has been shown to work for those students who do well at school. Although, parents and teachers might be thankful when a child receives a diagnosis as they feel they now better understand what kind of problem the child has, some researchers caution that applying a medical term to a child’s problem can sometimes have a detrimental effect, as such a diagnosis runs the risk of becoming an easy ‘escape’, preventing the exploration of other contributory reasons. In addition to neurophysiological and medical factors, it is argued that it is also important to consider the student’s environment as something that might affect their mathematical ability (Rosenqvist, 2007).

Engström (2003, 2009) underlines that there is no real agreement or consensus on how a learning disability in mathematics might be defined on a neurophysiological basis. This serves as an argument against the concept of dyscalculia, and it is obvious that the preference in Swedish mathematical research is to
avoid applying neurophysiological and medical labels such as dyscalculia. Instead, other hypotheses and theories are put forward in attempts to explain the emergence of students’ difficulties with mathematics. Sjöberg (2006) begins his thesis by postulating that the reasons for students’ problems with mathematics may better be found in flaws in the teaching and in students’ level of application. Sjöberg carried out detailed case studies of 13 students who had learning difficulties in mathematics, documenting 40 lessons on video and conducting two in-depth interviews with each of the 13 students. This empirical study shows shows that there is a complex web of factors which may potentially affect student achievement. Sjöberg noticed that the students’ engagement in learning decreased when groups of students were too large, when lessons were missed and if there was a stressful work environment. Sjöberg claimed that there is no strong scientific evidence to support the use of the term dyscalculia in practice. Ostad (2001) points out that low achievement in mathematics can develop over the course of a student’s schooling. Low achievement is not necessarily related to an organically induced malfunction in the brain, but the chosen teaching approaches can be important for the students’ mathematical development. A study on the importance of their social network for students with difficulties in mathematics (Karlsson, 2009) revealed that social problems at home diminished the students’ ability to learn effectively in mathematics. As such, there is reason to assume that students’ social networks have a considerable influence on the development of mathematical difficulties.

The working hypothesis for this empirical study is likewise that, in most cases, socio-cultural factors in the school environment and the impact of the student’s own social network are the root causes of the student’s difficulties in mathematics.

**Theoretical framework for learning mathematics**

According to the so-called multi-factor model, students’ learning of mathematics depends on three main factors: the mathematical content, the student’s personality and the student’s social network (Magne, 2002; Engström & Magne, 2003). The first factor, mathematical content, refers to the abstract concepts introduced to students. Impaired learning of mathematics may be associated the mathematical content as, for students, the abstract nature of mathematics itself can be the source of the student’s difficulties. Another factor is the students themselves. We know that there is considerable variance in terms of students’ knowledge levels, and according to Magne, these differences depend on the student’s development to date. There are socio-biological limits to each student’s knowledge, but learning
is also affected by emotional experiences (Magne, 1999). A third factor, according to Magne, is the social network that surrounds the students in the school, as well as the social class and level of education of their parents. In practice, this encompasses the state’s rules and regulations such as the Education Act and curricula; the school itself and its specific norms and values; and life at home and among peer groups. Students belong to a complex social network and all these factors affect how and what they learn.

Another theoretical point that underpins the research questions by focusing on factors that affect the learning of mathematics, and even other subjects, at school is described by Illeris (2006). The first term is the learning process, where a driving force of psychical energy leads to students acquiring the content of the subjects they learn. This internal psychological process involves the mobilisation of the mental energy that learning requires. At the same time, Illeris asks why teaching does not always lead to learning. In other words, why do mathematical difficulties arise? Here Magne’s multi-factor model is useful, as it emphasises both the nature of the subject and the student’s ability to mobilise the energy needed to acquire knowledge (Magne, 2002). The second concept, the interaction dimension, affects the social aspects of learning (Illeris, 2006). This may include the student’s interaction with his or her environment, which can occur on different levels. For example, on one level there is the class or work group, while on another level, there is the society in which the student lives. Action, communication and collaboration are key elements in our relationship with the outside world, according to Illeris. The interaction dimension contributes to the development of students’ sociality; that is, their ability to engage with the outside world. Once again, there is a link to Magne’s multi-factor model which also shows how important the students’ social networks are in the development of good mathematical skills. Karlsson (2009) showed in his study that students’ social backgrounds are in many cases crucial to their ability to cope with their difficulties in mathematics. In this context, Illeris’ theory that the interaction with the environment is an important process for the individual student’s learning is important. The educational situation not only affects learning but is also part of it. Students with unstable social conditions have poorer opportunities to complete the learning process than those with a secure social background. Learning always takes place in the context of the external social environment (Illeris, 2006).

Seen from the theoretical framework for learning mathematics drawn from Magne, Engström and Illeris, a neurophysiological framework is too narrow to explain why some students have low achievement in mathematics. The framework from Magne, Engström and Illeris can, however, be combined with the descriptive terms SUM and specific SUM.
Methods

A quantitative survey was carried out in Swedish lower secondary schools with Grade 7-9 students in three of Scania’s municipalities in order to assess the numbers of students with SUM and specific SUM at the end of Grades 8 and 9.

The quantitative survey investigated students’ final grades in mathematics in spring 2009 in Grades 8 and 9 to identify how many students had not passed mathematics in the final examinations. Approval was obtained from the school administration in the three municipalities in Scania. The collection of grade data was carried out by administrative staff who provided written summaries of final certificates. High reliability has therefore been maintained (Bryman, 2002).

A qualitative study was also included in order to enable identification of the various factors that, according to the students’ explanations, lie behind the students’ poor achievement levels in mathematics. In this qualitative study, semi-structured interviews were conducted with some of the Grade 9 students from both groups: SUM and specific SUM. With help from the schools’ administrative staff, eighteen Grade 9 students were randomly selected to participate in interviews. Eight students chose not to participate, so a total of ten interviews were conducted with Grade 9 students instead of the planned twelve.

To highlight the background conditions for student achievement, the questions were divided into two sections. First, the questionnaire queried the student’s experience of mathematics teaching, possible influences on the student’s knowledge, and the approach and organisation of teaching. The second section, meanwhile, contained questions concerning the measures taken in response to the student’s low performance in mathematics and the student not achieving the approved level in Grade 8. It was also important to get a picture of the student’s performance in mathematics during previous school years and what steps had been taken to help the student.

Results of rating inventory

The first research questions regarding the proportion of students in Grades 8 and 9 in the three Scania municipalities who are classified as having special educational needs in mathematics and the number who have specific SUM can now be answered. In the academic year 2008/09, 9.4% of students in Grade 8 did not achieve a pass grade in mathematics, and 3.9% belonged to the specific SUM group. In Grade 9, 4.7% of students did not achieve a pass mark in mathematics, and only 1% of all students in this grade were classified as having specific SUM.

Municipality B had the highest proportion of students who failed to achieve a pass grade in the subject; namely 15.7%. It is of note that, in this academic year,
the municipality B cohort in question had been assessed using principles that differed from those usually used. One explanation given for this in personal conversation with heads of school administration was that the teachers in this municipality wanted to give Grade 8 students a mark that reflected the amount of work they would need to do to pass their examinations at the end of Grade 9. For these reasons, the data from Municipality B have been compiled separately from the remaining Grade 8 information as the result was not representative of the grade in question.

Table 1 summarises the frequency of specific SUM in the other Grade 8 and 9 classes.

Table 1: Frequencies of specific SUM

<table>
<thead>
<tr>
<th>Municipality</th>
<th>Grade</th>
<th>Number of students with specific SUM</th>
<th>Percent</th>
<th>Total number of students</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>8</td>
<td>4</td>
<td>1.4</td>
<td>285</td>
</tr>
<tr>
<td>A</td>
<td>9</td>
<td>3</td>
<td>1.1</td>
<td>274</td>
</tr>
<tr>
<td>B</td>
<td>9</td>
<td>3</td>
<td>1.0</td>
<td>315</td>
</tr>
<tr>
<td>C</td>
<td>8</td>
<td>3</td>
<td>1.6</td>
<td>189</td>
</tr>
<tr>
<td>C</td>
<td>9</td>
<td>2</td>
<td>1.1</td>
<td>173</td>
</tr>
</tbody>
</table>

Has the hypothesis that there will be very few students who can be categorised as having specific SUM been confirmed? About 1% of students were assessed as having specific SUM in the reported grades. This low proportion corresponds with the view taken by Magne (2006).

Results and analysis of student interviews

Among the ten Grade 9 students interviewed in the academic year 2009/2010, three were girls and seven boys. In Grade 8, five of them only failed to achieve a pass mark in mathematics (specific SUM), while the other five did not pass in mathematics, Swedish and English (SUM). None of the students interviewed were from an ethnic minority background. To address the conditions for failing in mathematics according to the students own explanations, I first provide a short summary of the interview findings in Table 2. Students’ names have been changed to preserve confidentiality.
Table 2: Summary of student interviews

<table>
<thead>
<tr>
<th>The student</th>
<th>Cause of difficulties</th>
<th>School actions</th>
<th>Frequent change of Teacher</th>
<th>Anxiety about mathematics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Anna</td>
<td>Reading difficulties&lt;br&gt;Bullying</td>
<td>Separate group</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Bertil</td>
<td>Lack of motivation&lt;br&gt;Frequent teacher changes</td>
<td>Separate group</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Christer</td>
<td>Personal problems in the family</td>
<td>Special teacher in class&lt;br&gt;Welfare officer</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>David</td>
<td>Reading and writing difficulties&lt;br&gt;Problems with concentration</td>
<td>Separate group&lt;br&gt;Special teaching</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>Emma</td>
<td>Extensive truancy and school fatigue</td>
<td>Separate group&lt;br&gt;Special teaching</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>Irene</td>
<td>Severe problems in the family</td>
<td>Separate group</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Mats</td>
<td>Lack of motivation</td>
<td>Separate group&lt;br&gt;Special teaching</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Nils</td>
<td>Extensive reading and writing difficulties</td>
<td>Separate group&lt;br&gt;Special teaching</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Ove</td>
<td>Extensive absenteeism</td>
<td>No action</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>Sven</td>
<td>Lack of motivation</td>
<td>Action programme</td>
<td>No</td>
<td>No</td>
</tr>
</tbody>
</table>

It is interesting that none of the ten students interviewed attributed their difficulties with mathematics to the level of abstractness or to neurophysiological reasons. All the students reported other reasons; for example, extensive truancy, family problems, neglectful teaching and dyslexia related issues. It was noteworthy that no student had been intensively tested in order to determine the nature and extent of his or her difficulties in mathematics. Only one student stated that the
school had developed an action plan as a result of his difficulties. However, eight of the ten students were in Grade 9 placed in special teaching groups where they enjoyed teaching that they found meaningful. One student remained in the class and occasionally got help from a visiting specialist teacher. Only one student rejected all supplementary, special educational help offered.

Half of the students reported that frequent teacher changes aggravated their situation. Seven students reported that there were problems with the working atmosphere among the students. They spoke of their own nervousness and anxiety before the test sessions, and of disorganised and noisy lessons that distracted them and made work difficult. According to the students themselves, this contributed to both initiating and aggravating their low performance in mathematics. Three students reported that their anxiety about mathematics made it more difficult for them to learn. None of the ten students stated reasons for mathematics difficulties that might arise from neurophysiological or medical factors. This may reflect that there are no adequate measures of or techniques to recognise dyscalculia problems available to the students. Nevertheless, the statements from the students definitely show that the students in both the SUM and specific SUM groups most commonly viewed the background for their problems in mathematics as derived from their social networks; something which also influenced the students’ motivation. A less common cause was reading and writing difficulties.

**Discussion**

Ratings were reported from all of the 762 students in Grade 9 and 474 students in Grade 8. Three municipalities with varying demographic profiles were chosen to investigate the level of students’ grades. Almost 6% of all Swedish students in grade 9 in the academic year 2008/2009 went to school within the three municipalities involved in this study, which provides a good basis for generalising the results. Given that only 1% of students in Grade 9 in this investigation belonged to the specific SUM group, it can be said that there are fewer students who have specific difficulties in mathematics than some proponents of the dyscalculia concept assert. Results from literature that the percentage of students with dyscalculia is 5 - 6 % (Shalev, 2004) are not supported and seem unrealistic in light of the results of this study when, in the three municipalities included, only a few students in the grades studied have specific difficulties in mathematics.

The third research question relates to the reasons behind the students’ grades in mathematics, as the interview responses shed light on students’ own perceptions of the causes for their difficulties. Both in SUM and specific SUM students’ views, social factors influenced their mathematics achievement levels. These views
are in accordance with Illeris (2006), who underlines that the interaction with the environment is an important process for the individual student’s learning, and that educational situations do not only affect learning but are also a part of it.

As a factor influencing student achievement, social network includes effects of social class status and educational achievement of parents and the effects of conflicting norms in the student’s environment. Social network also includes the effects of curriculum implementation and school organisational design. Another subsystem is the school itself, with its various norms, values and skills. The systems are dependent on each other and affect the results obtained in different ways. Studies show a strong link between students’ performance in mathematics and their social background. According to the Swedish National Agency for Education’s statement on ethnic minority students, in mathematics education, students with highly educated parents score higher. Students who consistently show poor performance in mathematics tests often have parents with unskilled occupations. Parents’ social status affects the social networks that influence students’ mathematical development.

The analysis of the semi-structured interviews adds to the discussions on the hypothesis that the students’ social network is by far the most influential factor when it comes to explaining student’s low achievement in mathematics. The analysis shows that, according to students’ own perceptions, this hypothesis seems valid. Both the students in the SUM and the specific SUM group regard the surrounding social factors as having an impact on the problems behind their low performance in mathematics. No systematic differences could be found between the perceptions in the two groups of students.

The school and the larger educational system make up an important part of students’ social network. However, the support provided to students with low achievement in mathematics was limited, according to the student interviews. The support paid little attention to the explanations for the students’ low achievement levels, and, when given, focused narrowly on special teaching of mathematical content, without taking into account broader aspects such as motivation and social network, or other problems such as reading and writing difficulties. Only one student stated that the school had developed an action plan as a result of his difficulties.

The study reveals that among the complex reasons for low achievement in mathematics put forth in the literature – stretching from complex social networks to dyslexia - the students themselves think that social background plays an important role in students’ work and how they learn. Social problems at home may have diminished the students’ ability to assimilate mathematics teaching, and social context in school may have had an important impact. Despite the knowledge
from national and international studies that social and environmental factors are reflected in school performance, research in this area is scarce; for instance, the question of mathematics performance as a class issue has received little attention.

**Recommendations for educational research on mathematics difficulties**

Empirical educational research that can clarify the reasons why students are performing poorly in mathematics should be given a more prominent role in the general didactic debate. To try to put solutions in place, we need to further analyse the explanations behind the rise in low achievement in mathematics, as seen in Sweden.

The Swedish National Agency for Education recognises in its report on special support in elementary school that differences in goal achievement between schools are largely explained by factors such as students’ gender, parental education levels and ethnic minority backgrounds (Skolverket, 2008). In particular, parental education is found to have a clear relationship with the student’s academic achievements. Teacher competence is described in the research review as the single most important factor for successful educational outcomes.

In the author’s opinion, more research is needed on students with special and specific educational needs in mathematics. If teaching should be tailored to the needs of the students in order to help them manage their everyday lives and their future professional tasks, a more flexible curriculum could be recommended, as it allows an adaptation of the subject matter to the students’ level and needs. Also, learning mathematics need not only be about learning the skills to secure future prosperity. My own hope is that all students, including those who struggle with the negative effects of their social network, can experience the joy and fascination of the creative designs of mathematics.

**References**


Svensk sammanfattning

Särskilt och specifikt utbildningsbehov i matematik


Nyckelord: dyskalkyli, matematiksvårigheter, särskilt utbildningsbehov i matematik (SUM), specifikt utbildningsbehov i matematik (specifik SUM), socialt nätverk.
“There Is Always Something New After Nine”
- Action Research as a Mode for Teachers to Develop a Tool for Analysing Their Pupils Numeracy Skills

By Jónína Vala Kristinsdóttir, Dóróþea Reimarsdóttir, & Hafdis Guðjónsdóttir

Abstract
This article reports on a school-based collaborative action research project that focused on the development of an assessment tool for teachers to use when analysing children’s numeracy skills and mathematical thinking. The research was conducted by a mathematics teacher educator and a professor in special education, both from the School of Education at the University of Iceland, alongside a special education teacher at a compulsory school (municipal schools for children aged 6-16). The project began with the special education teacher’s enthusiasm for supporting children who struggle with mathematics and interest in helping classroom teachers learn how to assess children’s mathematical thinking so that they can use this information when planning lessons. In the cyclical research process, the two teacher educators served as critical friends and researchers by supporting and collaborating with the special education teacher on systematic documentation of her experiences. The educators were responsible for the broader data collection through narratives and their analyses. The purpose of the research was to create a collaborative study and collect and present data about the teacher’s work as she created, used and developed an assessment tool. The goal was to understand and
gain knowledge of how the teacher developed the assessment tool and how she presented it to the classroom teachers. The findings indicate that the assessment tool has the potential to be useful for assessing children’s mathematical thinking and for teachers’ continuing development in working with children.

*Keywords*: numeracy, special education, action research, narratives, teacher development, assessment tool.

**Introduction**

In this article, we focus on how, during the action research period, a special education teacher developed her strategies for responding to pupils who had difficulties with mathematics. One of the goals of the action research was to present the methods that the teacher had developed for working with young children, their classroom teachers and their parents to build a community of learning in which the resources that children bring to school are respected.

The principle that guides the special education teacher’s work is that all children should be supported in learning mathematics in a manner that facilitates their understanding and in making sense of what they are learning. Through her 30 years of experience as a mathematics teacher, she has learned that procedural learning is not sufficient for children; it is neither sufficient for those who struggle the most with mathematics nor for other children. She trained as a special education teacher by virtue of her growing interest in supporting children who find mathematics difficult. During her special education studies, the teacher began to develop a framework through which teachers could learn how to understand children’s thinking about numbers and calculations which she used in interviews with children who scored low on national tests in Grade 4 (9-year-old pupils).

When using the assessment tool in Grade 4, the special education teacher understood that it was important to support children at an early stage, and she then began to seek additional tools that could help her analyse Grade 1 children’s mathematical understanding. During this process, she collected data that helped her rethink her approach and develop the assessment tool in her work with children. Through professional collaboration about special education needs in mathematics, the teacher educators had followed her work and served as critical friends in this cyclic process. When she started to develop the new assessment tool, they offered to support her in structuring her work and adopting the action research cycle, thus changing the focus of her professional developmental work to become a research project.
The purpose of this action research was to create experiences and knowledge to support improvements in the participating teacher’s practice and to communicate interesting findings to other teachers and parents. A specific goal was to create and develop an assessment tool. Therefore, the action research was guided by the following research questions:

- How can a special education teacher develop and use an assessment tool to create an individual educational plan?
- How can a special education teacher and classroom teachers create a learning environment that supports all children in learning mathematics?
- How can teachers involve parents in the support of their children as they develop numeracy?

In the following sections, we will discuss the theoretical background that the special education teacher referred to during her work. We will also introduce the methodological approach and the methods we used in the research. We then sketch out and discuss our findings.

**Theories of young children’s development of mathematical thinking**

The theoretical framework that guided this action research builds on sociocultural theories and research findings on children’s mathematical development. The framework is based on Hiebert et al. (1997), who emphasised that learning mathematics with understanding is important because things learned with understanding can be flexibly applied, adapted to new situations, and used to learn new things: skills that are crucial in a changing world.

Piaget’s theories of child development, as well as the belief that knowledge does not exist in a static form but is an ongoing constructive process characterised by an origin and its development (Piaget, 1969), guided the construction of the framework of the assessment tool. Piaget’s descriptions of children’s developmental stages, particularly the fact that children accomplish these stages differently, were used when the assessment tool was constructed.

Piaget’s theories of how children assimilate and accommodate knowledge, as well as how children gradually assimilate information regarding reality by modifying it and integrating it into an existing schema, are the foci of the analysis of children’s development. This filtering or modification of the input is called assimilation, whereas the modification of internal schemas to fit reality is called accommodation (Piaget, 1969). It is important for teachers to recognise that
children are active learners who need to be respected as such as they develop mathematical thinking.

It is also important for teachers to be aware of children’s developmental processes and understand how they are shaped by the cultures in which they live. Socio-cultural theories are historically linked to the work of Vygotsky. According to Vygotsky (1978), children solve practical tasks with the help of their speech, as well as their eyes and hands. The child plans how to solve the problem through speech and then implements the prepared solution through overt activity; thus, there is a dynamic relation between speech and action.

The zone of proximal development (ZPD) is one of the concepts of Vygotsky’s theory, probably best known amongst teachers, and it is important that they are aware of the manner in which they can support children to develop their mathematical thinking. The zone of proximal development defines those functions that have not yet matured but are in the process of maturation; that is, functions that will mature tomorrow but are currently in an embryonic state. It is important that teachers understand that the actual developmental level characterises mental development retrospectively, whereas the zone of proximal development characterises mental development prospectively (Vygotsky, 1978).

Sarama and Clements (2009) have argued that the development of numerical skills begins early and when children begin school most of them have mastered the basic counting skills. Children are able to produce the standard list of counting words in order, understand that the order is constant and to connect this sequence in a one-to-one manner to the items in the set being counted. Children have also understood that the last counting word indicates the amount of the set or the cardinality of the set, and furthermore, understand that one can also count things that are not visible or cannot be touched. Additionally, it is likely that they are aware that the counting of objects in a set does not set conditions as to where one begins the counting and that every number includes the number before. This understanding means that those children who have difficulties with the idea of cardinality when they begin school need support in developing their mathematical thinking and it is important for teachers to be aware of how they can support them.

As the understanding of our number system is of special importance for children’s further learning, teachers need to have knowledge of how children’s thinking regarding the place value of the base ten system develops. In the early grades, children develop an understanding of the fact that items can be grouped to make a larger unit, and in a written multi-digit number; the value of a digit depends on its position because different digit positions indicate different units. Moreover, teachers must be aware of how the grouping of larger units, such as
making groups of tens and hundreds, can help children develop place value concepts; for example, that the position of a digit indicates its value. Children move from a counting-based view of numbers to a view based on grouping and place value, in which the development of these concepts is intertwined (Baroody, 2004).

Based on theories of children’s developmental processes and socio-cultural theories regarding mathematical learning, the special education teacher searched for tools that may be helpful in assessing young children’s mathematical thinking. She built on her former experience of framing an assessment tool based on the Cognitively Guided Instruction (CGI) project (Carpenter, Fennema, Franke, Levi, & Empson, 1999) and the findings of Reys et al. (1999) on children’s understanding of numbers and operations where she had adapted her framework to the goals of the National Curriculum Guidelines for Mathematics (Menntamálaráðuneytið, 2007).

The Early Numeracy: Assessment for Teaching and Intervention (the Mathematics Recovery Project, MRP) (Wright, Martland, & Stafford, 2006) was one of the tools she investigated. She finally decided to use it as an aid in framing an assessment tool for herself and classroom teachers in Grade 1 (six-year-old pupils). The mathematics recovery assessment aims to provide extensive and detailed information regarding a child’s numerical knowledge. This includes obtaining detailed information about the child’s current numerical strategies and knowledge regarding number words and numerals. To obtain the required details of the child’s numerical knowledge and strategies, an assessment interview is conducted. The interview is videotaped, and the outcomes of the assessment are then determined by an analysis of the interview. The teacher uses a flexible approach, posing additional tasks and questions on the basis of the child’s initial responses to the tasks. Following the assessment, a profile of the child’s knowledge is created.

The teacher’s development was cyclic, as she constantly reflected on her work and sought research on children’s development of numeracy that could help her in developing her practice. Thus, she was adapting the process of researching her own practice.

**Action research – research methods and data sources**

Action research differs from other research methodologies in many ways. Practitioners are involved as researchers where the purpose of the research is to improve their practice, and multiple cycles of inquiry are used; therefore, the outcomes of earlier cycles influence subsequent thinking (Quigley & Kuhne, 1997). Action research is always small-scale research because the intention is to improve or change practices and report on the development. The research topic arises from practitioners’ questions, and the goal is not to report facts of knowledge.
(Baumfield, Hall, & Wall, 2013). The research discussed in this article is school-based collaborative action research conducted by a special education teacher, mathematics teacher educator, and a professor of special education (Schmuck, 1997). The purpose of this action research, conducted by one teacher researcher and two academic researchers, was to gain experiences and knowledge of how we can collaborate on doing research and at the same time support improvements in the participating teacher’s practice and communicate interesting findings to other teachers and parents. The main goal of the research was to document how the teacher created and developed an assessment tool.

Because it is often unrealistic for teachers to pursue research in addition to their full-time work, and most teachers are not trained as researchers, the teacher educators served as critical friends with whom the special education teacher collaborated on data collection and analysis (Robinson & Lai, 2006). In action research, multiple techniques, data collection, and analysis are interwoven throughout the process as participants document their experience (Guðjónsdóttir, 2011; McNiff & Whitehead, 2002; Sagor, 2005). In this research, data consist of documents from the teacher in the form of her assessment tools, individual plans, case writings, and minutes from meetings. Furthermore, the findings from the data are incorporated into ongoing cycles of this research. This research is a responsive project, given that the special education teacher is responding to the need for an assessment tool that examines pupils with difficulties in mathematics (Schmuck, 1997).

The special education teacher developed and used the assessment tool and, at the same time, collected data on this process. She interpreted and evaluated her findings and, according to them, she changed and modified the assessment tool. In addition, she collaborated with the classroom teachers in her school during the continuing development of the tool and modified it according to their suggestions. The action research system of cycles helped the special education teacher to build on research findings as she created and developed the assessment tool.

Conducting thematic analysis, the co-researchers analysed the data from the process, unfolding special events and scenarios that were vital to understand what was occurring as the assessment tool was created and developed. Findings from action research are often presented as narratives, located in the context of the evolving experiences of those involved (Heikkinen, Huttunen, & Syrjälä, 2007). Thus, the co-researchers again combined the data into narratives that are reported here to provide a picture of how the special education teacher, reflecting in collaboration with others, leverages her past experience as she shapes her present practice. The narratives are based on interviews with the special education teacher and the notes she wrote throughout the project.
Findings

In this section, we report on our findings. Through narratives, we provide an account of how the special education teacher learned from her previous experience of developing an assessment tool to assess fourth graders’ thinking regarding numbers and calculations. Developing a tool to use with first graders, she chose to use The Early Numeracy – Mathematics Recovery Project (MRP) as well as her experience gained through the development of the first assessment tool.

At first, the special education teacher attempted to use the assessment tools from MRP. Based on her earlier experiences, the teacher then began to develop and adapt these tools to the conditions in her school. This narrative is an example of her use of the assessment interview:

I was working with Guðrún, a six-year-old girl. We were working on the unknown factor, and I tell her that I have seven red cubes and place them under a sheet of paper. I ask her to turn around so she cannot see what I am doing. Then, I add some yellow cubes and tell her that now there are ten cubes under the sheet and ask her if she can tell me how many cubes I added. She responds right away: “Three”. I ask her how she figured that out. “Because I know that seven plus three is ten”. I continue by telling Guðrún that I now have 12 red cubes, and altogether there are 15 cubes. I then ask Guðrún how many yellow cubes I placed under the sheet. She responds: “Twelve [little hesitation], thirteen, fourteen, fifteen” (stretches her fingers for 13, 14, and 15). Looks at her fingers and says: “You put three”. Next, I show her the number sentence 16 - 12 and say: “Can you read this?” Guðrún nods and reads: “Sixteen minus twelve”. Then, I ask if she can find the solution for me. She stretches up all her fingers, and for each number she mentions, she bends one finger at time. “Fifteen, fourteen … six”, then she stretches up two more fingers, “five, four. It is four”.

In her analyses, the special education teacher realised that Guðrún’s response was in coherence with what she was familiar with from her earlier work, and correlated with the early numeracy project. Guðrún knew that seven plus three is ten and could use her knowledge to determine how many cubes added to seven would make 10. When the pupil does not know the facts regarding the numbers used, she counts her fingers to help her keep track of her counting. The pupil is also capable of reading the number sentence 16 - 12 and because counting backwards is difficult, the fingers again support her in keeping track of her counting.

The special education teacher recognised that Guðrún solved this practical task with the help of her speech, as well as her eyes and hands. The teacher’s
knowledge of children’s development, as theorised by Vygotsky, regarding the manner in which the child plans to solve the problem through speech and then by means of overt activity, as well as the teacher’s knowledge of the developmental stages of numeracy skills, all contributed to her analysis of the child’s knowledge of numbers.

Another example from an assessment interview opened the special education teacher’s eyes to children’s interpretation of place value:

I was working with Siggi, who is a seven-year-old boy, assessing his numeracy. He mastered counting from whatever number until he reached 109. Instead of saying 110 he said: “1000”. I asked him to count again beginning from 96 and the same thing happened, he said: “109, 1000”. I asked if he remembered that he counted: “7, 8, 9, 10” and he nodded and said: “Yes”. I continued and told him that we do the same with 107, 108, 109, 110, but now he didn’t agree and said: “No, there is always something new after nine”. Reflecting on his response reminded me that this is a common response, and many children have difficulties naming the right number after 109; most often they mention 200 or 1000. However, his response, that there is always something new after nine, got me wondering if children at this age have figured out a system as they count but when they come to 110, their system doesn’t work because then they have to continue to say: “hundred and …”.

As can be observed in this example, Siggi was assimilating his counting into his schema of the place value system, but was still not able to filter in the teacher’s explanation and thereby modify his understanding of the place value. The special education teacher’s knowledge of Piaget’s theory of assimilation and accommodation helped her interpret Siggi’s explanation and helped her relate to former experiences of similar explanations offered by other children.

When the assessment tool was initially offered to the classroom teachers, we noticed hesitation and found that they preferred to rely on the special education teacher to perform this type of evaluation. However, in the second cycle, one classroom teacher agreed to conduct assessment interviews with her first grade pupils. This teacher found it rewarding to interview her pupils and was surprised at how much she learned from interviewing them with the support of the assessment framework. Consequently, she slowly adopted the systematic reflection on mathematical interactions that focuses on pupils’ learning and understanding of processes as well as on one’s own interaction behaviour. We also learned that the special education teacher’s objective to support her colleagues in adopting this manner of working as professionals was developing as she observed the
classroom teacher’s growing interest. After observing the special education teacher, the other first grade teachers at the school decided to use the framework to interview their pupils. The classroom teachers first observed the special education teacher conduct interviews two or three times before using the framework themselves.

The results from the interviews are now used as the primary tool for teaching in the first grade classrooms. The special education teacher supports the classroom teachers in working with the children who need special attention and works with these children in a group of three or four. The focus is on counting, acquiring the cardinality principle, and understanding place value and the base-ten system.

The parents are also supported in working with the children at home. The parents are encouraged to play games in which counting is used and to take advantage of every opportunity in daily life to exercise the children’s counting skills. These games may include playing card, board and computer games in which counting is needed to solve problems. The following narrative highlights what the special education teacher learned regarding parents’ support when conducting an assessment interview.

One day I was working on counting with Pétur, who is an active six-year-old boy. I begin by telling him that there are five red cubes and four yellow cubes under my sheet of paper. Then, I ask him if he knows how many there are in total. “Nine”, responds Pétur quickly. When I ask him how he knows that he replies: “Because five plus five is ten and minus one is nine”. I continue working with him and tell him that there are nine red cubes under my sheet and six yellow and ask him if he knows how many there are in total. Again he responds quickly: “Fifteen”. I ask again how he knows that. “If there were 16 altogether and there were 10 red cubes instead of nine it is just minus one”. Next, I ask Pétur if he can read a number sentence to me and then find the solution. I show him 17 - 14, and he reads: “Seventeen minus fourteen is three”. I ask him how he found that out and he replies: “I just took four from seven”. I put nine cubes on the table and tell Pétur the number as I place a sheet of paper on top of them. I then take four cubes away and ask him if he knows how many are still under the sheet. He responds right away: “Five”. I ask him if he can tell me why he is so quick finding the answer. Pétur replies: “Because five plus five is ten and this is not quite the same, but still four plus five”. I ask him why he is so good at mathematics and he responds: “Because my mum and dad are always making problems for my brother and me”.

"There Is Always Something New After Nine"
As is illustrated in this narrative, the work the parents have done with their children pays off. Not only can the children solve problems at home, but they can also transfer their skills and confidence in working with numbers to other situations.

The third cycle is evolving, and the special education teacher is now developing a framework for intervention to support children with difficulties in learning mathematics. She plans to work with children in developing her work and then introduce her framework to classroom teachers and support them in working with these children to ensure that they develop their mathematical understanding.

Children who struggle with mathematics in school are deprived of the opportunity to learn mathematics in a manner that is meaningful to them. It is the school’s responsibility to support these children in developing their understanding and facility to use mathematics in their lives.

Conclusions

The participation in this collaborative research and the subsequent writing-up of the work has prompted us to reflect on our understanding of children’s development of mathematical thinking. The special education teacher has reflected on how young children develop their sense of the base-ten system and how she can support this development. The title of the article: “There is always something new after nine” refers to the manner in which children develop their own schema of place value. The teacher’s understanding of how children gradually assimilate information regarding the natural numbers and then modify their internal schemas to fit the structure of our number system has supported her in noticing how difficult it is for certain children to accommodate their internalisation of place value to their own schema. The teacher has recorded examples of children’s conceptions of the base-ten system that help her reflect on the means by which she can support children who have difficulties with these transformations. It is important to be aware of the children’s understanding at this stage of their development because it forms a bridge between early and later forms of their development in mathematical thinking. The results from Häggbom’s (2010) longitudinal research on children’s number sense cohere with the teacher’s experience in that young children’s number sense predicts their later understanding of fractions.

The research findings from the Ellemor-Collins and Wright (2007) research on children’s knowledge of the sequential structure of numbers also support this special education teacher’s experience of the difficulties children have with naming the numbers that follow nine and counting by tens. The numbers from 11 to 19 are also difficult for children and, according to Fuson et al. (1997) and Geary (2000), the same applies for children in other countries who speak languages of
European origin in which one places the unit before the tens, such as in fourteen. It is important that teachers are aware that writing 41 instead of 14 is not necessarily a sign of the child reversing the numbers; it may well be a sign that the child has not accommodated how we write numbers in his or her schema.

The main goal of school is to provide pupils with a learning environment that is respectful, caring, and safe. The school and the teachers are responsible for acknowledging and responding to individual differences and providing opportunities for all pupils to learn and succeed (OECD, 2009). The special education teacher in this action research project is concerned about her pupils’ understandings of numeracy; to respond to their individual differences, she uses an assessment tool that can help her capture these differences and create a learning environment that considers them as she plans and scaffolds her teaching.

The special education teacher realises that, if children’s everyday life is separated from what they learn in school, they may not make the connection needed between schoolwork and everyday work. By inviting the parents into a partnership, the teacher and parents, in collaboration, can help the children make the necessary connection to render mathematics practical.

As can be observed in this article, narratives can be one means to collect data for professional development; they can help with professional and practical reflections, pushing thinking regarding the work in an unforeseen direction and toward new learning (Attarda, 2012). Through action research methodology, practitioners, in collaboration with researchers, can analyse, evaluate, and make meaning of their authentic narratives. In addition to helping practitioners in their professional development, narratives can also support them as they present new knowledge in relation to their practice.

This research originated and developed through a teacher’s developmental work within her school. Collaborating with teacher educators in theorising and writing about her practice with the intention of making her developmental process public has resulted in this ongoing action research as an illustrative example of how action research can be the bridge between pedagogical developments and pedagogical research. According to Norton (2009), pedagogical developments include activities that have practical focus and may be presented as examples of appropriate practice. Pedagogical research has a more theoretical focus and is a more formal inquiry with accepted research methodologies. The former targets the generation of practical information that teachers may find useful in their daily practice; the latter targets the generation of theories that may work within schools but are not likely to have effects on school culture if they are not presented to teachers in a manner that is accessible to them. Our goal with this collaborative research was to build a bridge between theories that concern mathematics teaching
and learning and this teacher’s practice. Teachers are engaged in working with children and often they may neither have the time nor the experience to research their practice and write about their work. Thus, collaboration between teachers and educational researchers in researching practices is essential and is something for which teachers must be allotted the necessary time.

Cochran-Smith and Lytle (2009) emphasised the importance of teacher initiatives in research and in their everyday work at school. Practitioners are deliberate intellectuals who constantly theorise practice as a part of practice itself. The goal of teacher learning initiatives is the joint construction of local knowledge, the questioning of common assumptions, and the thoughtful critique of the usefulness of research generated by others, both inside and outside the contexts of practice.

This special education teacher has been consistent in her work in attempting to improve her practice and in seeking research on children’s development of mathematical thinking, which can assist her in helping children who struggle with learning mathematics. The teacher has also assisted other teachers and her pupils’ parents in supporting children in developing their mathematical understanding. The teacher’s collaboration with her colleagues and with her pupils’ parents has resulted in improved practice within the school and a supportive learning environment.

References


**Dansk resumé**

"Der kommer altid noget nyt efter ni"

- *Aktionsforskning som en måde, hvorpar lærere kan udvikle værktøj til at analysere deres elevers talforståelse*

Artiklen fortæller om et skolebaseret aktionsforskningsprojekt, som havde til formål at udvikle et undersøgelsesværktøj til lærere til brug i deres analyse af elevers talfærdigheder og matematiske tænkning. En speciallærer i grundskolen, en underviser på matematiklæreruddannelsen og en underviser på speciallæreruddannelsen stod for forskningen. Projektet udsprang af speciallærerens ønske om at støtte børn, som kæmper med matematikken, og hendes interesse i at hjælpe klasseundervisere til at lære, hvordan de kan undersøge børns matematiske tænkning og inddrage resultaterne af undersøgelserne i deres undervisningsplanlægning. I den cykliske forskningsproces havde underviserne på læreruddannelsen rollen som kritiske venner og forskere. De støttede og samarbejdede med speciallæreren om en systematisk dokumentation af hendes erfaringer og var ansvarlige for den bredere dataindsamling, for brugen af narrativer og for analyser. Målet med forskningen var at udforske og diskutere samt at udvikle og anvende det omtalte værktøj til undersøgelse af børns talfærdigheder og matematiske tænkning. Resultaterne af aktionsforskningen tyder på, at værktøjet er anvendeligt til at belyse børns matematiske tænkning og til den professionelle udvikling af lærernes kompetence i arbejdet med børn.

Nøgleord: talbehandling, talforståelse, specialundervisning, aktionsforskning, narrativer, lærerudvikling, undersøgelsesværktøj.
Mathematics Difficulties and Classroom Leadership

- A Case Study of Teaching Strategies and Student Participation in Inclusive Classrooms

By Maria Christina Secher Schmidt

Abstract

This article investigates possible links between inclusion, students for whom mathematics is very difficult and classroom leadership through a case study on teaching strategies and student participation in four classrooms at two different primary schools in Denmark. Three sets of results are presented: 1) descriptions of the teachers’ classroom leadership to include all their students in the learning community, 2) an account of how learning communities are produced by stated and practiced rules for teaching and learning behavior, and 3) descriptions of the classroom behavior of students who experience difficulties with mathematics. The findings suggest that the teachers’ pedagogical choices and actions support an active learning environment for students in diverse learning needs, and that the teachers practice dimensions of inclusive classroom leadership that are known to be successful for teaching mathematics to all students. Despite this, it is shown to be hard for the teacher to determine how students who experience difficulties with mathematics understand the underlying meanings of the tasks involved. The students’ possible difficulties are not apparent because the students seem to do the best they can and in the process, they look just like their peers. They fit in. As a consequence, the teacher does not adjust the teaching to the particular students.
**Keywords**: classroom leadership, classroom management, difficulties, inclusive mathematics, teaching strategies.

**Study context and focus**

Since April 2012, Danish legislation has redefined special needs education so only students needing support for nine hours or more a week can be counted as receiving special needs education (Undervisningsministeriet [Danish Ministry of Education], 2012). A review of the *Effekt og pædagogisk indsats ved inklusion af børn med særlige behov i grundskolen* [Effect and educational methods involved in the inclusion of children with special needs in primary school] finds that “all schools in Denmark are working toward developing a more inclusive school” (Dyssegaard, Larsen, & Tiftikci, 2012, p. 27, own translation). The political aim is that 96 percent of students in public schools should be included in general education in 2015. For mathematics teaching, the new legislation means that from now on students who previously received extra support in specially designed learning spaces will take part in general classroom teaching. Since all students still have a right to teaching that meets their needs, the change means that teachers of mathematics must now develop a classroom culture that effectively includes students in diverse learning needs.

Locating the research within an interpretivist tradition of constructivism (Ferguson, 2009) and using qualitative methods and a case study design, this investigation explores patterns in four mathematics teachers’ pedagogical choices and actions to answer:

1. What kind of teaching strategies are used to include students in the learning community?
2. What kind of participation strategies do students for whom mathematics is very difficult practice in the classroom?

The term *teaching strategies* points to the teacher’s behavioral, relational, and learning leadership in the classroom. The concept of *student participation strategies*, meanwhile, relates to the students’ practical sense that orients them towards appreciated norms in the classroom. The aim of the investigation is to contribute to a theoretical as well as a practical understanding of inclusive mathematics classrooms.
A pedagogical approach to learning difficulties in mathematics

This paper focuses on inclusion of students in learning difficulties. I say ‘in’ instead of ‘having’ or ‘with’ to stress the theoretical point that learning difficulties are not (simply) a developmental disorder that is inherent in the individual child, but (also) something created in the social context (Ferguson, 2008; Heyd-Metzuyanim, 2013; McDermott, Goldman, & Varenne, 2006; Roos, 2013). My approach relies on a social constructivist perspective and as McDermott and colleagues describe:

Who gets called LD [learning disabled in math], when, by whom, and with what results is organized by demographic and political conditions. LD is less a kind of mind, and more a method for differentiating people and treating them differently. (McDermott, Goldman, & Varenne, 2006, p. 16)

At the same time, in this study mathematics is viewed as life skills that stress mathematical literacy and employ the “math-hole” metaphor (Lindenskov, 2006, p. 77ff). Instead of imagining mathematics as a building that prescribes levels, the metaphor helps imagine mathematics as a landscape with mountains and valleys, and teaching as a way to guide students through this landscape. When you enter the mathematical landscape no matter who you are, you will always encounter problems in your understanding of the landscape. It is only when these problems seriously overwhelm students that they stop learning about the landscape – one could say the student is stuck in ‘a hole’ and cannot get out without help. The fact that mathematics is seen as a landscape means that whenever a student falls into ‘a hole’ – encounters difficulties – there are multiple pedagogical ways to cope with the situation. This approach means that in principle, students in difficulties, like other students, can participate in and learn from problem solving, discussions, and other activities. They should not primarily or only be involved in training basic skills.

To understand such learning difficulties, it is also necessary to focus on students’ affective and social contexts, which, in some cases, are the cause of (or contributing factors to) their problems. Sometimes the difficulties are caused by the curriculum, the teacher or the school as much as by the learner. In such cases, the ‘medical’ or ‘deficit’ model of special educational needs fails, because it looks for solutions in the wrong place. (Ernest, 2011, p. 24)

This perspective for explaining students in mathematical difficulties reflects a shift in international research from perceiving problems as being located in the
individual, to seeing difficulties in mathematics learning “as arising out of the interaction between learners and their learning environments” (Ernest, 2011, p. 5).

The notion of inclusive education is fueled by political arguments, for example, the Salamanca statement (UNESCO, 1994), and is transformed into a pragmatic rationale on the realization of inclusion (Carrington, 2006; Ferguson, 2008; Hedegaard-Sørensen, 2010; Tetler & Baltzer, 2011). These recent discussions and definitions still align well with the research of Ferguson (1995), where she describes:

*Inclusion is a process of meshing general and special education reform initiatives and strategies in order to achieve a unified system of public education that incorporates all children and youths as active, fully participating members of the school community; that views diversity as the norm; and that maintains a high quality education for each student by assuring meaningful curriculum, effective teaching, and necessary supports for each student.* (Ferguson, 1995, p. 286)

This study focuses on the pedagogical practices adopted by teachers of mathematics with a view to creating an environment that supports and facilitates both academic and social-emotional learning; in this study, these practices are labeled ‘classroom leadership’. The notion includes a wide range of teacher behaviors, such as the development of relationships in the classroom, organizing a learning environment that supports students’ opportunities for participation, developing students’ emotional and social skills and self-regulation, and the use of interventions that help students with behavioral problems (Evertson & Weinstein, 2006).

Findings from a systematic review (Schmidt, 2013) show that the development of relations in the class, as well as the way the teaching is organized and carried out, can help students participate. In this review classroom leadership includes learning, relational, and behavior leadership. The three dimensions of classroom leadership overlap in practice as a teacher will move among them allowing them to support and influence each other.

**Methods and methodological approaches**

The case study investigated mathematics teaching in four classrooms at two different primary schools that teach Grades 1-3. The participants were purposively selected because all four math teachers had received specific training in working with students in math difficulties as part of a municipal project on early math intervention, consisting of one-to-one teaching sessions with teacher-student oral communication on students’ strategies, conceptual understanding, and problem treatment. The teacher training included a 48-hour in-service course. As supported
by the Danish Evaluation Institute (Danmarks Evalueringsinstitut, 2011, p. 46), it is assumed that such specific knowledge would provide the teachers with more resources, skills, and engagement in promoting the inclusion of students in math difficulties than teachers without this qualification.

The two schools studied are located in the same municipality, where the socioeconomic conditions, specifically income and education levels, are generally favorable compared to the rest of the country. Most of the students live with two adults, although not always with both of their parents, and most of the parents are employed and have attended post-secondary education. The public schools have classes from preschool to 9th grade, and each has approximately 800 students, 60 teachers, and four pre-school teachers.

This paper presents data from both schools introducing Laila, who has been a math teacher for 26 years, Magnus, who has taught for 18 years, Irene (14 years), and Ann Louise (four years). In each class a comparison student (not in math difficulties) was selected along with two focus students assessed by the math teacher to be in math difficulties in terms of the metaphor about “math-holes”. The teacher made the assessment by observing the students’ behavior in the classroom and with the aid of a screening test. The test both establishes the student’s current proficiency and examines the student’s attitudes to math.1 The focus students selected are not diagnosed as being in other difficulties; thus they can be expected to have especially good chances of being included in mathematics contrary to students who have grown up under less favorable socioeconomic conditions and who are diagnosed. All the teachers and the focus students are ethnic Danes.

Such extreme case selection makes it possible to learn something valuable from exceptional surroundings that can be relevant in more typical settings (Flyvbjerg, 1991; 2006; Neergaard, 2007).

Data sources

The main data sources were four teacher interviews (one with each) and observations of 35 mathematics lessons. Individual interviews were conducted with the eight focus students and four comparison students and 83 student essays about their views on math were collected (from all but two of the students in the four classrooms).2 Two types of observations were made. One used an observation guide that resulted in student participation profiles using The Student Membership Snapshot (Ferguson, Rivers, Lester, & Droege, 1995; Tetler, Ferguson, Baltzer, & Boye, 2011), which is designed to provide a picture of the whole classroom while focusing on how a particular student in learning difficulties fits as a learning member of the classroom.3 The other type of observation involved video recording
every lesson using two cameras. One pointed at one of the focus students; the other captured the whole class. The choice of teaching situations to transcribe was made along with an initial analysis, since data collection and analysis always proceed together in qualitative methods. The transcripts made it possible to attend to certain things by simplifying the activities investigated, and therefore no transcription is ever complete (Lindwall, 2008).

Analysis

In the first stage of analysis all the video recordings were viewed (some parts several times), with systematic searching for recurring themes and items of interest, including note-taking and transcription of some sections of the tapes in detail. The data analysis was conducted with the knowledge that:

‘Reality’ is always already interpreted. Thus data never come in the shape of pure drops from an original source; they are always merged with theory at the very moment of their genesis. (Alvesson & Sköldberg, 2009, p. 58 - original emphasis)

With this perspective in mind, several careful readings of all of the data were conducted to identify how the data related to what was expected based on both a theoretical understanding of the issues and a systematic review of the dimensions that promote inclusion when teaching mathematics. Questions that were considered included: How do the teachers think about students in math difficulties and inclusive education? What kind of participation strategies do the students bring to bear? What kind of feelings and attitudes do students express regarding math teaching? What kind of teaching strategies are used? Findings from a systematic review (Schmidt, 2013) showed that teaching strategies can be analytically divided into learning, relational, and behavior leadership. These three notions provided a framework or structure for examining the data in the current study to understand what was happening in the classrooms.

The second stage of analysis focused on the videos of two of the classes to determine whether a theoretical perspective (Brousseau, 1984; 1997) might help to explain particular events and situations. Brousseau’s notion on “the didactic contract” was used with the same intent as Yackel & Cobb described:

There is a reflexive relationship between developing theoretical perspectives and making sense of particular events and situations. The analysis of the particular constitutes occasions to reconsider what needs to be explained and to revise explanatory constructs. Conversely, the selection of particulars to consider reflects one’s theoretical orientation. Thus, particular events empirically ground theoretical
constructs, and theoretical constructs influence the interpretation of particular events. (Yackel & Cobb, 1996, p. 459)

The approach taken was that practice is influenced by many rationales, and that a teacher’s actions must be understood within the particular social and institutional conditions of the concrete situation. This required that the researcher put herself in teachers’ places on the grounds that: “we must posit that understanding and explaining are one” (Bourdieu et al., 1999, p. 613 – original emphasis). At the same time, it was challenging to create a certain distance from which to see teachers, students, and teaching as a research entity – as an ‘object’ constructed in relation to specific social, economic, cultural, and institutional conditions that are historically conditioned. In addition, the social world includes everyday life experiences and understandings about whatever is to be investigated. The pedagogical field, including the researcher, students, and teachers, uses spontaneous notions to describe what is at stake (Larsen & Brinkkjær, 2003; 2009). In other words, ‘doxa’, which is the experience by which “the natural and social world appears as self-evident” (Bourdieu, 1972, p. 164), dominates, and this is crucial to what it is possible to say and do. One way to break with the ‘natural’ assumptions and try to show their ‘culturality’ is by observing the categories used in the study’s observations. This has required reflection upon the normative underlying basis and continuously paying attention to which examples are referred to as participation, learning, inclusion, learning difficulties, and so on.

Results

In the following three sets of results are presented: 1) descriptions of the teachers’ teaching strategies to include all their students in the learning community, 2) the learning community produced by stated and practiced rules for teaching and learning behavior, 3) the participation strategies of students in math difficulties.

The teachers’ teaching strategies

The four cases illustrate both similarities and differences in how the four teachers practice inclusive classroom leadership, and the following will illuminate patterns that reveal the teachers’ pedagogical choices and actions as they set up an active learning environment for students in diverse learning needs.

Regarding producing a learning community, the meta-study (Schmidt, 2013) showed that instead of asking so many questions, teachers should spend more time inviting students to explain what they are thinking while the teachers practice transformative listening. This kind of learning leadership is, to some extent,
practiced in the four classrooms of this study. For instance, all four teachers use math manipulatives and concrete materials in their teaching. The teachers’ intention is to make students reason mathematically and not merely memorize algorithms, as in this way students understand different dimensions and connections in the mathematical landscape. During the interviews all four teachers underlined the importance of including different representations of mathematics because otherwise students may only learn mechanical ways to solve problems. The teachers organized their teaching so that students were actively involved in the learning processes, and they strived to engage all the students in dialogue. As a consequence, the teachers seldom lectured.

Laila illustrated this approach to learning culture in discussion-based mathematics by wanting lessons to be exploratory and emphasizing that the teaching of mathematics can have a creative dimension. She wanted students to work together and engage in plenty of discussion. In the interview she gave an example that made her smile: The students in first grade had to learn about geometric edges (e.g., a square has four edges), so she looked in the textbook for ideas but she wanted to do something different because she found it too boring to count edges in pictures in a book. Instead she cut a length of ribbon, and tied it together so the students could make their own squares at their desk by each using one hand to make four corners, forming a square with four edges. In the interview she explains:

Then I said: ‘Can you make other kinds?’ Well, they could also make eight edges. Then some of them figured out that if they used their fingers they could make more edges. It went brilliantly; they were totally occupied with this piece of ribbon and the fact that they could make edges. It was quite magical. … Then suddenly some of them went over to another table and someone says: ‘We’ve made one with 70 edges’. I looked at these 70 edges, and they were extremely proud of it. I said to them: ‘What does this look like?’ Then, almost in unison, they say: ‘A circle’. They had done mathematical reasoning! They had discovered that a circle consists of many edges, an infinity of edges. It was just so great.

Laila often replied to students with a question like “How do you know that? What did you do that led you to this result?”

The next excerpt illustrates how the three dimensions of inclusive classroom leadership overlap in practice. Magnus used a mathematical game in third grade and his behavior leadership told the students which norms for participation were acceptable, including the fact that it was legitimate to speak about how they felt about math. This ‘permission’ fostered a sense of security in the students, which illustrates his practice of relational leadership. Finally, it was typical of his learning
leadership to use the very last minute of the lesson on something that related to the overall content of the day: multiplication.

Magnus says to the whole class, “Now you have to listen. We’re coming to the end of the school day. Listen carefully: This Jubii extra booklet you have received – you will soon be asked to put it into your box and then to pack away your pencil case. We have math again tomorrow morning. Listen carefully”.

There is a little rattling with the pencil cases, but no one speaks so he continues, “This is the end of the day and the sixth lesson of the day, and I can see that some of you work super-fast and that some of you are getting a bit tired, so we’ll give it full attention tomorrow morning when we do math again, right? Hey!” Two students have gotten up from their seats, but sit down again, without further comment from Magnus. “We’ll start with these two desks; you can go and drop your book in the box. ‘Go’ to you! Meanwhile, the rest of you can pack away your pencil cases. I’ll go around and collect the papers with multiplication tables”. Magnus continues to follow up with small instructions about which students should get up next.

When all the students are standing behind their chairs, Magnus asks them to find their favorite multiplication table. The teacher selects a few students and has a ‘badminton match’ with them across the classroom, explaining, “You have to shoot the numbers across the network and the other student has to immediately shoot the next number back”. Two students are taking turns saying the numbers in the two-times table out loud. A lot of students are eager to get picked for the next ‘match’.

When Yosef is selected he hesitates, and Magnus says, “What’s the worst thing that can happen?” Yosef shakes his head as if he does not know.

Magnus suggests, “That you get stuck?” Yosef nods and the teacher carries on: “You know what – then it’s nice to have such a good class, right?”

Yosef enters ‘the game’ and the teacher’s head turns from left to right as if he was watching a major badminton match – following the words with his head. The students are smiling. Some choose to say one of the tables backwards, and on one occasion the teacher decides that the new pair have to do it backwards.

In every lesson I observed, Magnus invited students’ ideas about the worst thing that could happen in mathematics classes. He made it clear that making mistakes is part of doing math, and he often stated that it is important to dare to participate. For example, he said, “What I really liked was when you were unsure, you took control and didn’t get stuck”. On another occasion when a student did not want
to answer, he said, “What’s the worst thing that can happen? Making a mistake?” The student says “yes” to this, and Magnus continues, “Oh you see, in math you have to make mistakes”. Magnus did not push the student any further, but asked someone else instead. He addressed the importance of making mistakes, but the class discussion did not involve collaborative analysis and the revision of flawed solutions.

All the teachers pointed out that noise is appropriate when students are working, and they wanted to encourage students to experiment, ponder, and figure out results on their own and together. Magnus explained that he wanted the class to value the time it took to consider, and he disapproved of students just shouting out results before everybody had time to think.

The teachers seldom planned to differentiate the methods and content, but the student-teacher dialogues revealed differences regarding goals and expectations for each student. In particular, Laila and Magnus did not focus their feedback on evaluating performance as right or wrong, believing instead that a teacher should appreciate the student’s effort to follow the teaching and then talk about “how do we find something that is more correct?”

Ann Louise attempted to highlight the purpose or intention of the teaching task by talking with the students about the plan for the day and the goals for the week. She used pictograms to show students what kind of organizational structure she wanted for their work. The most transparent framework used appeared in Magnus’s practice, because just as Magnus wanted to know what was going on in the students’ heads, he also wanted the students to know what was going on in his. Specifically, he wanted the students to be aware of his expectations for how they should behave and communicate in the classroom.

From the interviews I recognized that the teachers viewed students in math difficulties as being cautious and quiet, and they wanted to encourage and acknowledge students’ efforts to participate. For instance, Ann Louise said in the interview that she wanted the class culture to accommodate student differences so that it was legitimate for different students to receive different kinds of assistance, like a calculator or manipulatives. In relation to academic topics, the place value and the names of numbers often challenged students, but according to Ann Louise, the knowledge that they were allowed to look at the 10-times table, have a number board, use their fingers or sometimes a calculator really helped them. She also thought it was important to make the students contribute to class dialogue and discussion, and for students in difficulties to experience success – things they were able to manage. She wanted others in the class to see that students in difficulties were competent in some things so they are not seen as the poorest at math. Her experiences showed her that most of the students in difficulties found it
The teacher Ann Louise found it meaningful to work with concrete materials; and because geometry is something they can manage most of the time, Ann Louise said she tried “to give them some successful experiences in geometry”.

In her interview, Irene described students in math difficulties as “very cautious, and they don’t dare to engage in a task if they are not one hundred percent sure what to do”. The main project that students did during observations in Irene’s second grade was working with the position system, that is, knowing about four-digit numbers and place value. They had to cut numbers with a pair of scissors into groups of thousands, hundreds, tens, and ones and then put the four strips on top of each other in the right places in order to produce specific four-digit numbers from a number board. It is like a kind of bingo, and they are told to take turns putting the number on the number board. One of the students in difficulties, Sandra, asked the classmate next to her what she should do, even though it was the same assignment as the previous week. The classmate showed her what to do, but she only got as far as making a number in the number table when the teacher came to her desk and guided her through the task:

“Is it right?” Sandra asks, still apparently unsure.
“Yes, nice,” Irene confirms. “Can you try to say the number, Sandra?” Sandra pouts and says, “I’m not good at big numbers”.
But Irene pushes her a bit by saying, “No, but try anyway. What’s the first number you have found?”
Irene makes the number more manageable by separating the strips of paper into 6000, 500, 40, and 7, and in the end Sandra reads 6547.
Irene then leaves the desk and Sandra smiles and says to herself, “Now I have made a number”.

Irene explained in the interview that students in difficulties are easily distracted and she used reverse psychology, or what she calls “opposite psychology”, by asking them what they find easy because

some of the students who find it difficult to get started, it’s also because they don’t believe in themselves. So, [it’s about] confidence … They resist a little and so I start by saying, ‘Well, let’s just try to see the task. It’s easy enough; you can easily do this one. So let’s try one more time’. I guess I’m very appreciative and [supportive of their efforts]. And so all the time I make sure that they are thinking: ‘Well, I can do this’.

In different ways all the teachers attributed value to and made efforts to encourage positive student relations, for example, when Irene wanted the lunch break to feel like a community, or when Magnus explained what a good working partner had
to do, or when Laila said that the worst thing that could happen in her classroom was students withdrawing from the community. She wanted them to go into “the fight together”. The data indicate that teachers’ understanding of students’ learning needs and different ways to use their knowledge about students’ emotions regarding math teaching seemed to affect students’ participation in the teaching. The four teachers had an approach to education based on ‘kind demands’. This means that students are met with care and trust. Relations are created that convince students that the teachers believe they are capable of learning and participating.

In sum, the case study indicated that the teachers’ beliefs, pedagogical choices, and learning environment orchestration constituted a form of classroom leadership. The learning leadership of the four teachers was nourished by dialogues and cooperation between teachers and students and among the students. Relational leadership was oriented toward creating a sense of security, and supporting a classroom culture in which it was legitimate to run risks when answering questions without worrying about whether the answer is correct. Behavior leadership valued ‘working noise’ together with concentration and norms that communicate that it is appropriate for students to spend time thinking about their answers.

Stated and practiced rules for teaching and learning behavior

Since the research focus is on how academic communities are created and what helps create a culture of mathematics in the classroom, this section explores the value orientations and evaluation practices of teachers of mathematics. It includes an analysis of how the practice of learning leadership contributes to the processes of inclusion and exclusion in math. Although the study had four teachers, in the interest of space only examples from Ann Louise and Magnus are used to show how understanding of the subject and choice of content and method can support or hinder patterns in the class that make it possible for students legitimately to receive help. In other words, the section explores the relationship between “the didactic contract” (Brousseau, 1984; 1997) and the learning leadership of the teachers of mathematics – but the use of the didactic contract does not imply that the whole theoretical framework of the theory of didactic situations has been imported. The didactic contract

constitutes the rules of the game in the classroom, rules that on the one hand frame the practices that emerge and on the other are regenerated and transformed by those very same practices. (Wedege & Skott, 2006, p. 41)
The term ‘contract’ should not be understood as an explicit and conscious agreement, but rather as a metaphor for the mutual agreements created through processes of socialization, with an impact on teachers’ and students’ expectations of how teaching and interaction in the class should take place.

When the structured observations were analyzed, the following temporal relations emerged as a percentage of all of the observed teaching in the four classes.

Table 1: Teaching orchestration

<table>
<thead>
<tr>
<th></th>
<th>Magnus</th>
<th>Ann Louise</th>
<th>Laila</th>
<th>Irene</th>
</tr>
</thead>
<tbody>
<tr>
<td>lead/demonstrate</td>
<td>22.4%</td>
<td>8.7%</td>
<td>19.5%</td>
<td>26.7%</td>
</tr>
<tr>
<td>lecture/tell</td>
<td>2.0%</td>
<td>2.2%</td>
<td>14.7%</td>
<td>6.7%</td>
</tr>
<tr>
<td>ask/answer</td>
<td>44.9%</td>
<td>30.4%</td>
<td>29.3%</td>
<td>None</td>
</tr>
<tr>
<td>support/supervise</td>
<td>14.3%</td>
<td>39.1%</td>
<td>36.6%</td>
<td>66.7%</td>
</tr>
<tr>
<td>observe/provide feedback</td>
<td>16.3%</td>
<td>19.6%</td>
<td>None</td>
<td>None</td>
</tr>
</tbody>
</table>

The category described here as ‘lead/demonstrate’ comprises situations in which the teacher initiated or explained a new task. The ‘lecture/tell’ category involved the teacher giving a somewhat lengthy explanation of a mathematical challenge. The category ‘ask/answer’ refers to situations in which a classroom dialogue between the teacher and the students took place. These three forms of organization are dominated by what Brousseau (1997) describes as didactic situations, that is, situations in which the teacher plays an important role, for example, by giving presentations, asking questions or providing explanations.

When the teacher moved around the classroom and helped the students while they worked on tasks individually or in groups, the activity was categorized as ‘support/supervise’, while the category ‘observe/provide feedback’ was used when the teacher evaluated the students’ work. It can be difficult to differentiate between these two categories, as the teacher’s support and supervision is also based on the teacher’s continuous observation. These last two categories are related to what Brousseau calls a-didactic situations. In these situations, the teacher plays a discreet role in the activity, and the goal is that students should take charge independently. They should not focus on the teacher’s expectations, but instead act in relation to the task at hand.

It is the combination of didactic and a-didactic processes that constitute Brousseau’s answer to the question: How can teachers ensure both that students learn and that what they learn is compatible with established understandings of the
subject? The combination provides a way to describe the compatibility between the students’ subjective knowledge, but also culturally developed and generally accepted understandings of the subject which they need to acquire.

When the time spent on activities categorized as ask/answer, support/supervise and observe/provide feedback is added up for these classrooms, it is clear that students are active, communicating, and solving problems for a considerable amount of the teaching time. One could conclude from this data that the teachers are practicing their expressed pedagogical ideal (that the students should relate actively to mathematical tasks in order to acquire an understanding of mathematics). The question is whether Magnus and Ann Louise actually establish an a-didactic game, or whether they are working in learning situations that have what Hersant and Perrin-Glorian (2005) call “a-didactic potential”. In the a-didactic game, the goal is that the students should act in and with a learning environment to devise solutions themselves. But if an environment does not provide feedback about whether a student has chosen a good strategy to reach a solution that works, there is a risk that acquiring new knowledge will remain a potential. For students in math difficulties, the didactic situations did not appear to create sufficient support to enable them to work without the teacher’s guidance in a-didactic situations.


The interviews with the teachers revealed how teachers wanted to affect students’ pre-understanding of these questions – although not so much what mathematics is, but what the teaching of mathematics is. Ann Louise, for example, said that she thought “mathematics is more fun [than Danish] because it’s possible to play and experiment more” and “it’s not just numbers, there is so much to do with patterns, forms and shapes. Geometry takes up a lot of time”. Magnus explained it this way: “Good teaching is teaching where you [the student] dare raise your hand, even if you aren’t sure. And where you feel that you survived even if you said something that was wrong”.

Regarding how students learn mathematics, Ann Louise said: “It’s really important that they express in words what they’re doing and what they’re thinking; that they don’t just sit there, doing exercises in a book; that they explain things to each other. I may [have tried to] explain something and there may still be someone who
says, ‘I don’t get it’. So sometimes I ask, ‘Is there anyone who has understood it who wants to try to explain it?’ And often the kids are actually better at explaining it to the other kids. I see that as a huge learning opportunity too”.

Magnus put it in another way: “It’s not enough for me to say that things are this way or that and to do exercises … All that about actually being able to see it in different ways – they don’t get that if I just say: ‘Now you are going to practice by doing twenty-five exercises’”. I query him further, “So it is important to use many different types of representation?” “Yes”, he replies, “otherwise you get at best something mechanical that can’t necessarily be used elsewhere … [you have to be able to] use it when you’re in a different situation. I have to ensure that they understand it”. Magnus elaborated this perspective in relation to why students should learn mathematics: “Mathematics is a tool you have to be able to use when you’re in the kitchen or in other contexts and later on in life. It must be a tool that means you will often guess approximately what the result could be”.

The observation data showed that the teachers valued working with metacognition. This occurred in part through dialogue about the teaching, as when Ann Louise dealt with the goals of the week and led a discussion about what the students learned; and also in part by making explicit what the students master, like at times when Magnus got the students to self-assess which times tables are easy/difficult and which tables they knew forwards/backwards. Magnus also worked with the metacognitive dimension when he framed discussions about students’ attitudes and feelings about mathematics by talking about “the worst thing that can happen”.

In sum, the stated and practiced rules of the game for teaching can be characterized through what the teachers acknowledge as ‘good’ learning behavior:

- Students who use mathematical concepts in their answers
- Students who explain the calculations they have used to arrive at their result
- Students who show an ability to meta-reflect on their acquisition of knowledge

At the same time as these norms unfold in the classrooms, the norm that dominates all the others is that students must be active participants in the teachers’ activities. The next section offers an interpretation of how this norm seemed to have a decisive influence on the participation of the focus students in math difficulties, which may further help in understanding the concepts of inclusive classrooms.
The participation strategies of students in math difficulties

One pattern in all four classrooms is that students in difficulties have a lot of interactions – both with peers and the teacher – and they need help several times during every lesson. All of the eight focus students tried to engage in the tasks, but they seemed to pay little attention to the didactic situations (especially class discussion and demonstration). Nevertheless, the focus students seemed motivated and eager to participate in the a-didactic situations, as three examples from three different classrooms show: 1) Heidi, who organizes numbers, 2) Ingrid, who makes geometric shapes, and 3) Alice, who looks as if she is having a conversation.

Students in second grade have groups of numbers organized into thousands, hundreds, tens, and ones. They also have a list of four-digit numbers. Their task is to recreate the four-digit numbers by selecting the correct strips of thousands, hundreds, tens and ones. Heidi spends the time on organizing and grouping the numbers by type instead of creating the four-digit numbers. Heidi puts the strips into four groups and all the numbers into sequence. In trying to keep the groups of numbers separate, she takes the numbers in and out of her plastic folder at least five times. It looks as if she is trying to find a way to keep the groups separate, but does not know how. The lesson is 20 minutes long and Heidi spends at least 15 minutes organizing. It looks as if she is doing the same as her peers, but she never produces a four-digit number.

Students in the third grade are making three-dimensional shapes by cutting shapes out of paper and folding and pasting them into a cone, cube, and so on. They are supposed to prepare a report that records the number of edges, surfaces, and corners. Ingrid is busy cutting and folding geometric shapes, and she tries to do the report part – she starts out by using her ruler to measure the form. She looks unsure and draws the shape but erases it several times and then picks up a new geometric shape. She makes four shapes during the lesson – two pyramids, one rectangle, and one cylinder. Ingrid picks up a new piece of paper every time she makes a geometric shape, so she is doing part of the task like the rest of the class although she does not produce any reports.

The students in third grade are working on number operations like adding, subtracting, and multiplying. They are supposed to be explaining to each other the way they figure out the different problems. Alice explains her calculating processes to Nadia as follows: “plus, plus”, but other than that they do not say anything else to each other. When they are supposed
to switch turns, Nadia asks, “Alice, are you looking to see if I do it the right way?” “Hmm,” says Alice. After a short time, Alice writes something on her own paper and then erases it again. It does not look as if her writing has anything to do with Nadia’s work. Later, when they are supposed to partner and practice saying multiplication facts to each other, Alice behaves the same way. She participates in the ‘dual circle’ in the sense that she stands in front of her partner, looks at her partner and smiles at the right time, but she does not say any words and does not really engage in the task.

In these examples the focus students participated and possibly learned some of the mathematical skills at hand. But how do we critically compare how the focus students understand the underlying meanings of the task compared to the majority of their classmates? Or their motivations and ability to use what they learned in their lives outside of school? And how can we understand why Heidi, Ingrid, and Alice seemed un-engaged in getting help from the teachers? Drawing upon different theoretical perspectives, possible explanations are offered for grasping some of what was going on.

Goffmann (2005) offers a way to interpret the students’ actions as creating a strategic social space where those involved attempt to portray a particular image that is shaped largely by dominant school discourses about what constitutes a ‘good’ student. In these four classrooms, the dominant (and appreciated) expectation is to be self-confident, communicative, and enthusiastic. Such actions can be referred to as “face-work”, with participants regulating themselves and managing their presentation in the classroom despite what they may be thinking or feeling. Originally the notion of “passing” (Goffman, 1963) referred to agents withholding information about themselves that was stigmatizing in order to gain acceptance. This was an intriguing way of describing interactions when analyzing students’ participation strategies; and in some ways, the focus students manage information about themselves that might reveal mistakes or ways in which they do not understand, which could lead to stigmatization, by mimicking the performances of their peers. But it is crucial to note that nothing points to the conclusion that students consciously intend to hide their difficulties or overtly try to sell a particular image. It seems more like an embodied “disposition” (Bourdieu, 1972) that is socially produced. This disposition to act like their classmates means the focus students offer the expected behavior – they fit in. This perspective can be related to the term “prestigious imitation” (Mauss, 2004), which describes agents who imitate actions that have succeeded before and that they have seen successfully performed by people in whom they have confidence or who have authority over them.
Another interpretation is that the students in difficulties could be practicing their competence – what they are good at and how and when they ask peers for support. Drawing on the notion of “an invisibility cloak” (Tronvoll, 2000a; 2000b), which hides difficulties and allows students to disappear into the children’s classroom community, one could suggest that students unintentionally cover up their difficulties through a) doing what they have mastered, such as organizing, folding geometric shapes, and having the ‘right’ attitude when working in pairs; and b) having solidarity with peers, especially empathy for others’ situations and support to and from friends. Once again though, it is important to emphasize that what the students are doing is more unconscious than purposeful. But even so, the strategy results in a situation where the teachers cannot easily see what students do not understand – the difficulties the students meet in the mathematical landscape are not visible to the teachers when they scan that landscape.

Another possibility is that the actions of students in difficulties could arise from cultural assumptions about learning that guide both what students do and what teachers see. Perhaps it is what schools, teachers, students, and their parents view as a ‘good’ student and what it looks like when you are learning. If the assumption is that students need to struggle to learn and therefore have nothing to be ashamed of, then difficulties would be an opportunity to move ahead in the learning game (Lindenskov, 2000). Brady (2011) discusses the comprehensive research regarding how struggles are often opportunities for students to think about underlying mathematical concepts. Her findings suggest that teachers “need support with envisioning how students’ errors can be productively used as springboards for inquiry in the context of class discussion” (Bray, 2011, p. 35). Studies have shown that teachers in China and the U.S. respond to errors differently (Schleppenbach, Flevares, Sims, & Perry, 2007). Some teachers in China strive to design lessons where students’ mathematical misconceptions are brought to the surface and examined openly to promote learning, with students revealing their struggles. In contrast, teachers in the U.S. tend to avoid discussions of students’ mathematical errors, and students and teachers alike place more value on the ‘quick learner’ or the student who ‘gets it easily’. Given such different cultural assumptions, it could be that it is not so much about difficulties being ‘invisible’.

Instead, the important thing is that the students do in fact learn much of what is intended. These teachers are sufficiently successful with all the students to ensure that the students who are still in some difficulty have a kind of ‘learning cloak’ around them in the sense that they are eager to participate, they are trying their very best, they do parts of the tasks, they do not disrupt the teaching, they get help from friends, and they have interactions with their teacher. In other words, they are so much into the learning game that the teacher might be led to
believe that they do not need to be helped or challenged in their mathematical thinking, to be asked how their current understanding of the subject connects with previous elements of content, and so forth.

**Discussion – teaching strategies and students’ participation**

This article focused on possible links between inclusion, students in math difficulties, and classroom leadership. The case study investigated strategies for teaching and students’ participation in four classrooms at two different primary schools during math instruction. These teachers use recommended practices for designing mathematics learning in order to assist learning. At the same time, the difficulties still being encountered by some of the students escape their notice.

The article argues that the mathematics teachers have skills and reserves of energy to promote many of the dimensions identified by international research as important for inclusive classroom leadership. First, in relation to behavioral leadership, the teachers appreciate that students think carefully instead of providing a quick answer. It is not whether the answer is correct in the first place, but that students dare to participate. Second, with reference to relational leadership, the math teachers create situations where students in math difficulties experience success and engage as part of the learning community. One strategy they use is to believe the focus students can do the tasks. The teachers also try to create caring relationships with their students, and among the students themselves. For example, the students are given opportunities to talk about feelings and attitudes toward mathematics. Third, in relation to learning leadership, the teachers’ goals are to create investigative dialogues in the classroom and provide opportunities for students to work autonomously with the subject through various activities in order to achieve an understanding of mathematical connections.

The article examined the way learning leadership is organized and practiced. The math teachers’ pedagogical strategies are a didactic contract that promotes establishing a-didactic situations – where a large part of the teaching time is taken up by students being active. For instance, teachers had students make geometric shapes, talk with a partner, or organize numbers. Study of the established norms shows that teachers expect students to explain their calculations and to use mathematical concepts in their answers. But for students in math difficulties it seems that these norms are less accomplished. The analysis suggests that during activity-oriented parts of the lessons, the mathematical understanding of the students is rarely examined and challenged by teachers, nor are connections to other activities verbalized explicitly.
The final section of the article illustrates that the teaching may impact students’ participation strategies: Students do their best to participate in the learning community. This participation results in behavior involving the students in difficulties striving to be ‘good’ students, who work on the part of the task they can manage – without causing trouble or disturbances for anyone else. In spite of the fact that these mathematics teachers are prepared specifically to work with students in math difficulties, the analysis suggests that overriding school assumptions still dominate largely unnoticed and unquestioned: Children are institutionalized to be students who are doing things that look school-like.

When I shared preliminary results with the four teachers at a de-briefing seminar, they talked about restructuring the pairings and groupings so that peers could provide better support for the students in difficulties and encourage them to engage in higher-quality learning and dialogue. First, they would reorganize where students in difficulties sat. But there could be other ways to reform practice as well. As the case study suggests, it might be possible to work on establishing pedagogical norms that portray learning difficulties as valuable. The signs indicated that students who were acknowledged when in trouble, prospered with smiles. While creating a sense of security for the students, this also supported a classroom culture where it was legitimate to run risks when answering questions – and so a practice that could facilitate diversity. This would mean that teachers not only articulate that it is okay to make mistakes in mathematics, but stress that struggle is a process that should be sought to gain new knowledge.

In other words, it might be possible to plan lessons so errors happen frequently for all students (and not just those who are in difficulties) so that everyone experiences math struggles. The subsequent discussion and investigation of the ‘mistakes’ could be an inclusive strategy that would impact students’ learning in mathematics. With reference to the pedagogical principle of seminars where research findings need to be discussed, Bourdieu once said: “Nothing is more universal and universalizable than difficulties” (Bourdieu & Wacquant, 1992, p. 218). If educators begin to think of the ‘good’ math student as someone who struggles and makes errors, perhaps many of the institutional practices in our schools need to change, and perhaps we need error-oriented classroom leadership.

Acknowledgement

I am deeply grateful for Dianne Ferguson’s comments and suggestions for improving earlier versions of this paper and for her linguistic assistance.
Notes

1 The math teachers identified the eight focus students’ math difficulties to be: Too thorough, slow-working, lacking confidence, difficulties getting started on their own, lacking solution strategies, difficulties in seeing patterns, reduced numeracy (e.g. place value, enumeration, names of numbers), requiring a lot of adult support, difficulties staying focused and concentrated.

2 The interviews were semi-structured and photos were taken from each of the classrooms depicting different kinds of tasks, activities, spaces, and materials. The teacher interviews concentrated on the teachers’ beliefs about and experiences with students in math difficulties as well as inclusive teaching strategies. The interviews with the students focused on the students’ beliefs, emotions, and attitudes regarding math teaching and learning. All 16 interviews were audiotaped and transcribed. The data from the students are beyond the scope of this paper, which focuses on the teacher interviews and the classroom observations. Students’ perspectives are addressed in another publication (Schmidt, 2015).

3 The idea is to provide a snapshot of the lesson, containing both a wide-angle view of the class in general and a close-up view of the student of interest. The overall questions guiding the observations were: “What is going on generally?”, “What is the focus student doing?”, and “Who is s(he) doing things with?” These questions were all transformed into specific categories, for example, “Materials”, which was further divided into text book, copy-sheet/hand-out task, concrete materials, manual aids, technical aids, game/play/physical activity, story/drawing, or other. Another category was “Content”, which was classified as number/algebra, geometry, statistics/probability, or other topics.

4 For each lesson the participation profiles were combined with video observations into integrated field notes. The complexity of the entire data set was reduced into a profile of each class including first the teacher’s voice, examples from the observations, and finally, the students’ voices.

5 Writing the four profiles and analysis in English rather than Danish made it difficult to use spontaneous notions and, in some ways, helped to create a certain distance from the people and situations of study.

References


Dansk resumé

Matematikvanskeligheder og klasseledelse

– *Et casestudie af undervisningsstrategier og elevdeltagelse i inkluderende matematikundervisning*

Artiklen undersøger relationerne mellem inkludering, elever i matematikvanskeligheder og klasseledelse gennem et casestudie, der sætter fokus på pædagogiske strategier og elevers deltagelse i fire forskellige matematikklasser på to folkeskoler i Danmark. Der præsenteres tre sæt af resultater: 1) Beskrivelser af matematiklærernes klasseledelse i forhold til inkludering af samtlige elever i lærefællesskaber, 2) Hvordan lærefællesskaber bliver produceret gennem italesatte og praktiserede normer for undervisning og lærefærd, 3) Lærefærden hos elever i matematikvanskeligheder. Casestudiet synliggør, at matematiklærerne anvendte inkluderende klasseledelsesstrategier, der er videnskabeligt anerkendte som succesfulde. Et andet fund er, at elever i matematikvanskeligheder i høj grad praktiserede en pædagogisk norm om at være aktivt involveret i opgaven. Elever i vanskeligheder gjorde det, som der forventes af en ‘god’ elev ved at anvende forskellige deltagelsesstrategier såsom at udføre ting, de var gode til, få kammersæthjælp og imitere de andre elever. Disse deltagelsesstrategier resulterede i, at matematikvanskelighederne kunne blive usynlige for matematiklærerne.

*Nøgleord: klasseledelse, læringsvanskeligheder, inkluderende matematikundervisning, undervisningsstrategier.*
Inclusion in Mathematics
- The Impact of the Principal

By Helena Roos

Abstract
The objective of this paper is to describe the means by which the principal affects the process of inclusion in mathematics from a teacher perspective. Two notions form the theoretical foundation: participation and inclusion. The participatory perspective is provided by Wenger’s theory of communities of practice (1998). When discussing inclusion, Asp-Onsjö’s (2006) notions of didactical, spatial, and social inclusion have been used. The results are presented in two parts: the first presents identified communities at the investigated school and the second identifies codes of impact pointing towards inclusion in mathematics. When combining the participatory and the inclusive perspectives, codes of impact regarding inclusion in mathematics in the different communities were identified. Although there are different codes of impact in the different communities, one can identify several recurring codes when investigating the impact of the principal. The most frequent is courses. Competence, didactical discussions and planning also recur in the different communities. Investigating these codes of impact, there appears to be a gap between the steering of the principal and what actually occurs. The results indicate that the principal’s impact on inclusion in mathematics in the realisation arena is relatively weak.

Keywords: communities of practice, inclusive mathematics education, organisation, principal, special education.
Introduction

Many Swedish students struggle with mathematics in school. Therefore, mathematics education in Swedish schools received additional funding between 2009 and 2011. The government invested a total of 352 million Swedish kronor in various projects. The latest initiative, ‘matematiklyftet’ (in English, ‘raising the standards in mathematics’) will be completed in 2016 and will cost approximately 649 million Swedish kronor (Department for Education, 2012). Despite this massive investment in improving students’ mathematical outcomes in Sweden over the past decade, there is minimal visible benefit, according to the latest TIMSS\textsuperscript{1} investigation (Mullis, Martin, Foy, & Arora, 2012). An increasing number of students do not achieve the national goals in mathematics (Department for Education, 2014). This finding has led to certain students being identified as having a need for special education in mathematics. Schools are approaching this problem by attempting to use ‘inclusion in mathematics’ as a method to improve all students’ knowledge in mathematics. Although the term inclusion is used, it can occasionally be unclear what this means in practice or for research. Therefore, there is a need for clarification of what inclusion in mathematics implies, and if and how it can help students who struggle with mathematics. The overall objective of this research project is to empirically investigate what inclusion in mathematics education can be and how it is possible to develop inclusive mathematics education, based on special education needs in mathematics (SEM). A key person in deciding how to address SEM in schools is the principal; this paper presents an analysis of the impact of the principal on the process of inclusion in mathematics from a teacher perspective.

According to Hattie (2003, p. 171), the principal can have an impact on student achievement through her or his responsiveness to the students and by creating a ‘climate of psychological safety to learn’. The principal can influence the climate and students’ responsiveness through the pedagogical environment at the school, which they have a duty to organise in such a manner as to promote learning. Occasionally, principals will reorganise the learning environment to improve learning, although there is evidence that this approach is not always successful (Larsson, 1998).

In relation to mathematics education, the type of reorganisation suggested often revolves around placing students in different ability groups (Boaler, 2008; Stigler & Hiebert, 2009; Wallby, Carlsson, & Nyström, 2001). Many Swedish schools use ability grouping in mathematics in a bid to help students achieve their learning goals. Nevertheless, some research suggests that organisational differentiation does not achieve the improvements in students’ knowledge development
that are expected by teachers (Boaler, 2008; Persson & Persson, 2011; Slavin, 1990). Educational differentiation and individualisation are complex issues that require more investigation. This investigation of the notion of inclusion in mathematics education may be one way to illustrate differentiation and individualisation.

In the overall research project, inclusion in mathematics education is investigated through observations, group interviews, and interviews with teachers and students. This paper focuses on the research question, ‘What impact does the principal have in the process of inclusion in mathematics?’ Data for the investigation were gathered from interviews with the principal and teachers, with a particular emphasis on the teacher perspective. In this paper, I use the notion ‘students in special education needs in mathematics’. This phrasing was chosen because it signals that it is not a deficiency within the student; instead, it is something the student can get in and out of (Bagger & Roos, 2015).

Theoretical foundations

The research project which is the impetus for this article is based on two theoretical perspectives: a participatory perspective using communities of practice (Wenger, 1998) and an inclusive perspective (Asp-Onsjö, 2006). Thus, the study is grounded in a social perspective on learning. The overall principle of this perspective is that learning is considered a function of participation (Wenger, 1998). Participation is to be seen as a process of taking part and is an active process that involves the whole person and combines the things the person is doing, such as talking, thinking, and feeling (Wenger, 1998). Participation ‘goes beyond direct engagement in specific activities with specific people’ (Wenger, 1998, p. 57). When people engage in actions and negotiate the meaning of those actions with one another, a practice is created. Hence, the practice resides in a community of individuals with mutual engagement. Members of a community of practice are practitioners who develop a shared repertoire, such as experiences, tools, artefacts, stories, and concepts. The joint enterprise keeps the community of practice together; its pursuit is a collective process of negotiation by the participants in the community (Wenger, 1998).

From a participatory social perspective, inclusion does not only imply being physically present in the classroom, but also being included in the mathematical practice of the classroom. This form of inclusion has no physical condition; it is imaginary. Asp-Onsjö (2006) uses the terms spatial, social, and didactical inclusion. Spatial inclusion refers to how much time a student spends in the same room as his or her classmates. The social dimension of inclusion concerns the manner in which students participate in social, interactive play with others. Didactical
inclusion refers to the manner in which students’ participation relates to a teacher’s teaching approach and the manner in which the students engage with the teaching material, the explanations, and the content that the teachers may supply to support the students’ learning.

As previously mentioned, the study’s broad objective is to empirically investigate what inclusion in mathematics education can be and how it can be developed using theoretical concepts. The concepts used are communities of practice (Wenger, 1998) and spatial, social, and didactical inclusion (Asp-Onsjö, 2006). By following this theoretical structure, Roos (2013) has identified three communities of mathematical practice regarding inclusion in mathematics from a teacher’s perspective. The first community of practice is the community of inclusive mathematics, which is a community where all three types of inclusion identified by Asp-Onsjö (2006) are present. Within this practice, teachers’ sensitivity, acceptance, and teaching approaches are central regarding inclusion in mathematics. The second identified community is that of the mathematics classroom, which focuses on didactical inclusion and where individualisation and teaching approaches are central. The third community identified is the community of special education needs in mathematics. Central for this community are terms for teaching and mathematical knowledge within spatial and didactical inclusion. Student co-decision is also evident in this community of special education needs in mathematics.

This paper’s objective is to analyse the impact of the principal using the above-presented theoretical concepts, the identified communities of mathematical practice, and concepts of inclusion in mathematics (Roos, 2013).

Method

In the overall project, the notion of inclusion in mathematics is being investigated at a primary school. This notion is viewed as a phenomenon. One approach to understanding a phenomenon is to use interpersonal methods. These interpersonal methods provide understanding through interpretation (Aspers, 2007). To understand the notion and process of inclusion in mathematics, the actions and processes at the investigated school are interpreted.

To find and analyse processes and actions, the researcher must interact with the people involved (Aspers, 2007). In the overall study, interaction occurs through interviews, discussions, and observations. The interviews were semi-structured, with overarching questions regarding mathematics teaching and inclusion. The discussions were more spontaneous, focusing on what was most relevant in a particular situation. Two types of observations were conducted: observing participation and participant observations. In observing participation,
the researcher is merely an onlooker. A participant observer, meanwhile, interacts with the respondents and is more actively involved in the research.

To be able to identify a process of inclusion in mathematics, a longitudinal study design was used. In this instance, it was of particular importance that the process of inclusion in mathematics be understood to assess the impact a principal can have on this process. This understanding is achieved using an ethnographic approach, in which the main data source is the interpretation of the interviews and discussions. Accordingly, as previously noted, this paper presents a teacher perspective regarding the impact of the principal on inclusion in mathematics.

Informal and formal interviews and discussions were conducted with the principal, teachers, and the remedial teacher at a large primary school; furthermore, mathematics lessons were observed and recorded. The school, Oakdale Primary School, is located in a suburb of a medium-sized Swedish town. The students who attend the school are 6-12 years old. Seven interviews and four discussions have been analysed in this paper and are the basis for the results presented in the next section. The interviews are with the principal (Conrad) and the teachers, Amanda, Ellie, Anna, and Barbara (three interviews and four discussions). The interviews were conducted over the period of a year. Amanda, Ellie, and Anna are primary teachers who work at Oakdale Primary School. These teachers teach mathematics, as well as other subjects, in lower primary school. Barbara is a remedial teacher in mathematics who primarily works with students in special education needs in mathematics. Conrad has been the principal of the school for one year; however, he has been a principal at different schools for the past eight years. Prior to that, he was a mathematics and science teacher in lower secondary school.

A qualitative, semi-structured approach was used (Kvale, 1996) in the individual formal interviews. The teachers and principal were invited to elaborate on their view on students in special education needs in mathematics and the factors they considered crucial for students’ participation in mathematics in school. In the informal interviews and the discussions, the focus was on the education of SEM-students; however, the talk was informal and unstructured. The interviews and discussions were recorded and transcribed in full.

Analysis

Static-dynamic analysis is a technique in which the researcher first codes the data using a code-scheme developed from the empirical material and theory. Saturation is achieved when no further material is considered likely to change the coding (Aspers, 2007). In this study, static-dynamic analysis was used to
find keywords, make codes, and create categories. In the analysis, the empirical material was labelled. This type of coding forms the foundation for creating new theoretical categories (Aspers, 2007). In the coding of the data, sub-codes were identified by the researcher. Furthermore, the identified sub-codes generated a few major themes in the data; this was performed with the help of a code-scheme (refer to Table 1). The code-scheme was developed and used in three steps. First, the empirical data were analysed with static-dynamic analysis, and several communities of practice were identified on the basis of keywords which pointed towards the same practice regarding mathematics at Oakdale Primary School. These code words, for example, ‘we’, ‘us’, ‘together’, indicated mutual engagement, shared repertoire, or joint enterprise, which are described by Wenger (1998) as three constituents of a community of practice. When these words occurred, a community of mathematical practice was considered to exist. The analysis was conducted in iteration during the data collection, and several communities of practice were considered to exist. The second step involved the application of the theoretical aspects of inclusion of Asp-Onsjö (2006) to the data. Three aspects, spatial, didactical, and social inclusion, were used as a lens, and several codes regarding inclusion in mathematics were found in the data. These codes were grouped into major codes. Finally, the major codes regarding inclusion in mathematics were categorised into the constructed communities of mathematical practice at Oakdale Primary School by identifying when and in what community of practice the codes emerged. Table 1 illustrates the code-scheme constructed by the first two steps in the analysis.

Table 1:
Code scheme used in the analysis.

<table>
<thead>
<tr>
<th>INCLUSION</th>
<th>SPATIAL</th>
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Results

The results are presented in two parts. The first part describes the identified communities of mathematical practice at Oakdale Primary School. The second part considers the impact of the principal in the identified communities regarding inclusion in mathematics.

Communities of mathematical practice at Oakdale Primary School

As a result of analysing the interviews, five communities of mathematical practice were constructed. Three of these were the same as those identified by Roos (2013) when examining the teachers’ perspective of inclusion in mathematics. The first community of practice is the community of inclusive mathematics (CIM). The principal, Conrad, is a member of this community because he wants the staff at the school to ‘use the resources in the class [room]’, and he also wants all pedagogues at the school to take responsibility for all of the students in the class. Anna, Ellie, and Barbara are also members. Amanda is not a member of this community because she is explicit in stating that she prefers the SEM students to be excluded from her class: ‘They had nothing to do with the math we were doing here [in the classroom]. They were not at that level at all’ (Amanda). The mutual engagement in this community of practice is the development of mathematics teaching for students in special education needs. The community’s shared repertoire is their experiences of special education needs in mathematics and the talk regarding how to help the students understand mathematics.

The second community of practice visible in the data were the communities of mathematics classrooms (CMC); because there are several different classrooms, there are several different communities. Conrad is a peripheral member of these communities, with an external perspective. Ellie, Anna, and Amanda are members of each their mathematics classroom community, and Barbara is a peripheral member of all of the communities, with an internal perspective. The mutual engagement in these practices is the mathematics teaching and learning needed for all students to meet the goals outlined in the curriculum. The shared repertoire is the talk about the mathematics teaching in the classroom, the curriculum, and the use of different teaching materials in the classroom.

The third identified community of practice was the community of special education needs in mathematics (CSEN). This community of practice arises from the existence of special education needs in mathematics at the school. Barbara is a central member of this community of practice. Anna, Ellie, and Amanda are members. Conrad is a peripheral member. The mutual engagement supports the students in special education needs and their development of mathematical
knowledge. The shared repertoire consists of the artefacts associated with teaching, such as materials, games, and tasks, as well as the individual education plans and their content.

The fourth community of practice is the community of mathematics at Oakdale Primary School (CMO). Conrad is a core member and the teachers are also all engaged in this community because they teach mathematics at the school. Conrad has reorganised the mathematics teaching to increase goal achievement and to use the teachers’ competencies at the school in the best possible manner to support learning. He notes that ‘we should do the same things … we should know what we are doing’, referring to the mathematical content of geometry and basic arithmetic. The subject meetings every third week are a means to strengthen the community. All teachers in mathematics at Oakdale Primary School are members of this community. The mutual engagement is the development of mathematics teaching at Oakdale Primary School. The shared repertoire is the talk regarding the mathematics teaching overall, the curriculum, and ‘pedagogical concerns [in mathematics at the school]’ (Conrad).

The fifth identified community of practice is the community of student health (CSH). This practice impacts the teaching of the SEM-students because the members of this community of practice are involved in the decisions regarding who should receive special education at Oakdale Primary School. The core members of this community are the remedial teachers and the principal, Conrad. Other members are the school nurse and the school psychologist. Helping the students in special educational, social, and physical needs constitutes the mutual engagement in this community. The shared repertoire is the ‘case management’ (Conrad), which involves a cycle with four steps. Case management is a ‘pedagogical mapping resulting in an individual education plan, then an evaluation of the programme, actions, and follow-up’ (Conrad). The teachers at the school are peripheral participants in this community of practice; however, at the same time, they can influence and be influenced by it because they address cases to this group, they address student health, and they participate in the writing of the individual education plans.

Thus, there are five visible communities of mathematical practice at Oakdale Primary School, when considering the principal’s influence: community of inclusive mathematics (CIM), community of mathematics classrooms (CMC), community of special education needs in mathematics (CSENM), community of mathematics at Oakdale Primary School (CMO) and community of student health at Oakdale Primary School (CSH). All these practices overlap and influence each other; however, there are differences in terms of participants, mutual engagement, and shared repertoire which influence the process of inclusion in mathematics at Oakdale Primary School.
The impact of the principal regarding inclusion in mathematics

As noted earlier, several sub-codes regarding the principal’s impact on inclusion in mathematics were found in the data. Examples of these codes are the responsibility, flexible solutions, and the development of mathematics at Oakdale Primary School. These codes were categorised into the five identified communities of mathematical practice at Oakdale Primary School. The three aspects of inclusion have also been considered in the categorisation.

**Community of inclusive mathematics – the impact of the principal**

This community focuses on how to involve students in special education needs in the mathematics taught in the classrooms. Spatial inclusion is a key initiative. Conrad is determined to change the prior school culture of excluding SEM students from the classrooms. He emphasises that the person responsible for the SEM students is not the remedial teacher; they are the responsibility of the regular mathematics teacher. Barbara (the remedial teacher in mathematics) still feels responsible for the SEM-students; however, she is eager to find flexible solutions in collaboration with the mathematics teacher. Here, didactical inclusion becomes visible: ‘students should feel that they are having the same educational experience as their peers [both inside and outside the classroom]’ (Barbara). Ellie says: ‘if everyone thought a little more inclusively and did not exclude all of the time, it would be easier for everyone in the school’. Both these teachers highlight the issue of spatial inclusion (or exclusion). This emphasis indicates that Conrad’s determination to change the excluding culture at the school has not yet been 100 percent successful. Barbara has a difficult time finding flexible solutions in the organisation, ‘it [to be flexible] fails because we have three [students] from the other class’. Both Conrad and Barbara discuss creating courses to provide students with ‘an intense period’ (Barbara) after which they could return to the regular mathematics classroom and be included, both spatially and didactically. Ellie also highlights this when she stresses, ‘it is good to have a session a week [with the remedial teacher] and for the students to improve; it gives synergies in the classroom’.

**Communities of mathematics classrooms – the impact of the principal**

In these communities, competence in the classroom is an issue. The teachers and Conrad appear to equate mathematics within a teacher certificate with competence in teaching mathematics. Conrad is struggling to gain as much mathematical competence in every classroom as possible, which is in accordance
with the teacher certificate that was introduced in Sweden in 2012. Ellie stresses the importance of having an education in mathematics to help all students in the classroom. Amanda feels that it is ‘hard to explain mathematics because I do not have all of these different ways to explain that ... a maths teacher has’. In this community, there is also the issue of having time to provide all of the students the support they require. Both Barbara and Ellie note that the mathematics teachers spend much time with the SEM students in the classroom, and this is at the expense of the other students. Competence is within the didactical inclusion: to have the tools to reach all of the students. Time is not within any form of inclusion but appears to be an influential factor.

Community of special education needs in mathematics – the impact of the principal

In this community, as well as in CIM, courses are discussed. Conrad notes, ‘We address it [special education in mathematics] in a good way because we offer [additional] courses’, and Barbara says, ‘we believe in that [courses]’. Ellie points to courses in addressing student needs in mathematics: ‘She [a SEM-student] needs this course’. Although courses primarily occur in grades five and six, Barbara wants to use them in all grades. Conrad discusses ‘cramming’ in connection to courses, whereas Barbara and Ellie stress the understanding and need to feel competent when attending the small group. Anna expresses a desire to have more cooperation with the remedial teacher. This desire is something that Barbara often addresses. She wants ‘to plan together [with the maths teachers] and think about what to do, how to help; how we use each other in the best way’, although it is difficult to find the time. Barbara says, ‘you can steal a moment’ in the morning drinking coffee, which indicates that it is difficult to find time and space for didactical discussions and planning. Both courses and didactical discussions and planning are within didactical inclusion.

Community of mathematics at Oakdale Primary School – the impact of the principal

The teaching of mathematics on an overall level is discussed in this practice. Conrad notes that ‘we should do the same things [in mathematics]’, which refers to one of the objectives of the reorganisation at the school. The main reason for the reorganisation is to increase goal achievement. Conrad also emphasises cooperation and the utilisation of competences. To develop mathematics (and science) teaching, the teachers have scheduled subject meetings every third
week. However, it is not always possible for them to adhere to this schedule. ‘It’s really, it’s all very well intended, but then sometimes it is … in the spring there are many [occasions] that it disappears actually’ (Barbara). The remedial teacher in mathematics is not present at these meetings: ‘we usually have meetings with the student health team [at the same time]’ (Barbara). Conrad and Barbara note the need for mathematical discussions; however, the other teachers do not. Barbara is eager to have more of these didactical discussions in connection with lesson planning: ‘we put our heads together’ to develop mathematics teaching at Oakdale Primary School. Didactical discussions and planning ‘are learning opportunities for me too’, says Barbara. Although Conrad and Barbara emphasise the importance of didactical discussions and planning, the data show that there are few opportunities available for the teachers and remedial teachers to have such discussions. Competencies and didactical discussions and planning are placed in the category didactical inclusion.

**Student health at Oakdale Primary School – the impact of the principal**

Conrad discusses the students in special education needs in mathematics in this community; specifically, what to include in, how to write, and how to implement the individual education plans.

‘The pedagogical plans also form the basis for one semester or for goal achievement in an academic year, for the individual student and the class. You do it once; you do it when you do the pedagogical planning. It is also the base when you write your action plans; you pick your goals … for the individual action plans from the pedagogical plan’ (Conrad).

These topics are placed within the category of spatial and didactical inclusion. From this community, there has been a proposal to use a specific material identifying students’ knowledge in mathematics. Courses are also discussed in this community.

**Summary of the principal’s impact on inclusion in mathematics at Oakdale Primary School**

The matrix in Table 2 presents a summary of the analysis, with major codes for the principal’s impact in the different communities regarding inclusion in mathematics.
Table 2:
The impact of the principal regarding inclusion in mathematics.

<table>
<thead>
<tr>
<th>Community of practice</th>
<th>Codes of impact</th>
</tr>
</thead>
<tbody>
<tr>
<td>Community of inclusive mathematics (CIM)</td>
<td>Responsibility</td>
</tr>
<tr>
<td></td>
<td>Flexible solutions</td>
</tr>
<tr>
<td></td>
<td>Courses</td>
</tr>
<tr>
<td>Communities of mathematics classrooms (CMC)</td>
<td>Competence</td>
</tr>
<tr>
<td></td>
<td>Time</td>
</tr>
<tr>
<td>Community of special education needs in mathematics (CSENM)</td>
<td>Courses</td>
</tr>
<tr>
<td></td>
<td>Didactical discussions and planning</td>
</tr>
<tr>
<td>Community of mathematics at Oakdale Primary School (CMO)</td>
<td>Competence</td>
</tr>
<tr>
<td></td>
<td>Didactical discussions and planning</td>
</tr>
<tr>
<td>Community of student health at Oakdale Primary School (CSH)</td>
<td>Specific material identifying students’ knowledge in mathematics</td>
</tr>
<tr>
<td></td>
<td>Courses</td>
</tr>
</tbody>
</table>

Although there are different codes of impact in the different communities of mathematical practice at Oakdale Primary School, one can identify several recurring codes when investigating the impact of the principal. The most frequent is courses. Competence and didactical discussions and planning also recur in the different communities. All of these codes refer to didactical inclusion.

Conclusion

The results show that the impact of the principal from a teacher perspective regarding inclusion in mathematics is different in the different communities of mathematical practice at Oakdale Primary School. The codes of impact show that there are different codes in the different communities. These codes do not always align with the teachers’ views. In CIM, CSENM, and CSH, courses are the focus; however, in CIM, courses are not considered a good solution at all grade levels. The organisation is an obstacle in relation to flexible solutions, particularly regarding spatial inclusion. Here, the impact of the principal on the level of teaching is weak. There is a need to include didactical discussions and the planning of mathematics in CSENM and CMO. Having an allocated time every third week for these discussions does not appear to work in practice, and the data show that these meetings actually do not occur every third week, despite the intentions.
Furthermore, because the remedial teacher in mathematics is not present at these meetings, it is difficult to discuss SEM issues and have didactical discussions with the remedial teacher. This finding indicates that either the intentions of the reorganisation are not yet reflected in the teaching of mathematics for SEM students, or new obstacles have emerged in the new organisation.

Does spatial inclusion in mathematics favour didactical inclusion and vice versa? In the case of Amanda, no, and in the case of Ellie, yes. This finding suggests that competence in mathematics teaching is important to achieve didactical inclusion. The reorganisation at Oakdale Primary School regulates the competencies according to the teacher certificate, which should promote inclusion in mathematics eventually. However, these data reveal difficulties in achieving flexible solutions regarding the teaching of SEM-students. In this case, courses and the schedule prevent the adaption of special education in mathematics to address current needs. After the reorganisation, the data suggest that there has been no noticeable impact on the process of inclusion in mathematics thus far. As Gadler (2011) concludes in her thesis, there is a discrepancy between the formulation arena, where the decisions are made, and the realisation arena, where the decisions are executed. In this case, the issue is inclusion in mathematics. The results indicate that the principal’s impact on inclusion in mathematics within the realisation arena is relatively weak. There may be various explanations for this. Examining the data, one explanation may be that, although the principal is a member of all of the communities of mathematical practice at Oakdale Primary School, he is a peripheral member in CSEN and CMC. These communities of practice are the communities that involve the students. The three communities in which Conrad is a more central member, CIM, CSH, and CMO, are of a more general nature with respect to inclusion in mathematics. This finding may be one explanation regarding why the influence is weak on the realisation arena.

Examining the three different aspects of inclusion in this investigation, didactical inclusion appears the most important. Spatial inclusion is often embedded in the didactical, with respect to obtaining access to the mathematics taught in the classroom. Social inclusion, meanwhile, is not apparent.

This investigation of the impact of the principal regarding inclusion in mathematics identified the major codes of impact and concluded that the impact of the principal from a teacher perspective is relatively weak. However, it also generated a number of new questions requiring closer examination: Does the absence of social inclusion have any impact on the process of inclusion in mathematics at Oakdale Primary School? Does the impact of the principal change over time regarding inclusion in mathematics? Will there eventually be an enhancement of inclusion in mathematics, when the utilisation of competences is optimal?
Notes
1 TIMSS represents Trends in International Mathematics and Science Study.
2 COP represents Communities of Practice.
3 The teacher certificate was introduced to ensure that every teacher in Sweden is properly trained for the subjects and grades they teach.

References
Svensk sammanfattning

Hur påverkar rektorn inkludering i matematik?

dominerande. Den spatiala inkluderingen är oftast inbäddad i den didaktiska med hänvisning till att få tillgång till den matematik som lärs ut i klassrummet. Social inkludering är inte synlig.

*Nyckelord*: praktikgemenskaper, inkludering i matematik, organisation, rektor, specialundervisning.
Student Equity vs Test Equality?

- Support During Third Graders’ National Tests in Mathematics in Sweden

By Anette Bagger

Abstract

This article throws light on the educational practice of teachers providing additional support to students during tests, more specifically during the national tests in mathematics for third graders in Sweden (hereafter called Ntm3) for the years 2010 and 2011. Both the test instructions and teacher talk related to these tests were taken into consideration. The results suggest that issues of equity and teachers’ agency arise when considering support. The dual purpose of the test, to evaluate the student and to evaluate the education, positions the teacher as both a test-giver and a test-taker and influences the discourse on support by ambiguity. I found that in such circumstances, when students’ equity comes into conflict with the test’s equality, the focus during the tests shifts from attention to learning to attention to controlling.

Keywords: national tests, mathematics, third grade, special needs, special support.

Introduction

This article presents results from a large scale\textsuperscript{1}, longitudinal ethnographic study of the reintroduction of Ntm3 (the national tests in mathematics for third graders; i.e., students usually aged 9-10) in Sweden. The results raised questions regarding
the support given to some students during the tests. National testing in Sweden can be seen as part of an international trend of a neoliberal education policy and governance of schools leading to a marketization of the school sector (Telehaug, Mediås, & Aasen, 2006). Values such as equality and social fairness are challenged by values related to consumption and competition (Hudson, 2011).

National tests in the third school year in Sweden were implemented in 2010 (Utbildningsdepartementet [Ministry of Education and Research], 2008) to monitor the national quality of education as well as the individual level of knowledge (Boistrup & Skytt, 2011; Regeringen [Swedish Government], 2006; Skolverket [Swedish National Agency for Education], 2012a). This included identifying students in additional needs earlier in their educational journey (Regeringen, 2006; Utbildningsdepartementet, 2008), an issue that was highlighted in the Government’s decision to implement the tests:

*The Government believes that it is important with early follow-ups and efforts to provide all students with the possibility of meeting the national targets and proficiency requirements. The school must have effective procedures for continuous monitoring and early appraisal of the students’ knowledge in order to identify students who have difficulty achieving proficiency.* (Regeringen, 2011, p. 3, own translation)

These students, in turn, require supportive actions by staff in order for the test to function as the above mentioned identifier; otherwise, it will be the support, or lack of such, that affects achievement rather than the student’s ability. The teacher is in charge of implementing supportive actions, with guidance from test instructions. The practice of giving support is understood as a discursive practice. As such it is situated in the context, culture and history of each classroom and school, as well as in the Swedish educational system in general (Skott, Van Zoest, & Gellert, 2011). In other words, giving support is an activity that stretches beyond the immediate situation in the classroom. How support can be talked about and how the students’ needs are met by the teacher are not merely choices made by an individual; they are procedures regulated by discourses in these different arenas. Earlier findings in a Swedish sample, of which this article’s sample is part, have shown how a discourse on testing coexists with a competitive discourse and a caring discourse during the third graders’ national tests (Silfver, Sjöberg, & Bagger, 2015). How these discourses influence and contravene one another affects the discourse on support in the classroom during the test. Thereby, they also influence the teachers’ practice of giving support. I draw on Hall (2001), who states that discourse:
... governs the way that a topic can be meaningfully talked about ... It also influences how ideas are put into practice and used to regulate the conduct of others. (ibid., p. 72)

As the teacher engages in the discursive practice of providing support, and rules and routines are put in place in the classroom, she is subjected to discourses and positions herself. Positioning is understood as

... the discursive process whereby selves are located in conversations as observably and subjectively coherent participants in jointly produced story lines. (Davies & Harré, 2001, p. 264)

The purpose of this paper is to seek the foundations of the discourse on support during Ntm3. The search for this is initiated in an investigation of how the support is framed in test instructions and teacher talk during and in interviews about the test situation. This is followed by an analysis of how the teachers are subjected and positioned within the practice of giving support in the test situation.

The national tests and the students in need of support

Most of the general background information about Ntm3 in 2010 and 2011 can be found in government documents, test instructions and the Swedish National Agency of Education’s database SIRIS². How the tests were carried out can be deduced from the tests themselves, instructions and field notes from eight classrooms at three schools in 2010 and 2011. This paper will also make reference to some of the earlier research on national testing and students in need of support. For the purposes of this article the status of ‘student in need of support’ is fluid (Silfver, Sjöberg, & Bagger, 2013). Educational needs might depend on the environment, mathematical content or emotional charge in the room or individual prerequisites.

During 2010 and 2011, the tests were carried out over a period of approximately one month. Instructions for teachers on how to provide support during the tests included an initial general introduction followed by instructions for specific parts of the tests that outlined how to provide support, what to explain and what aids to permit. Initially, the students completed a self-evaluation of their skills within the various mathematical subjects before continuing on to the actual testing of their knowledge within these subjects. The students took six individual tests and finished off with an oral group-test, in which they were mathematically evaluated as individuals³ (Skolverket [Swedish National Agency for Education], 2010b, 2011a, 2011c, 2011d, 2012a).
All students took the same tests with, often open-ended, questions covering a variety of mathematical skills, requiring the students not only to apply the mathematical methods that they had learned, but also to identify the exact nature of the problem at hand and, thus, what mathematical method to apply in the first place. The teacher told the students what to be aware of as a test-taker regarding the test, for example, how to understand questions or write answers. Important concepts and words were explained, often according to instructions. In addition, rules and advice were given on how to act during the test. This could include information about the desired behaviour during the test, how to ask for help or what to do when finished. These instructions took between 5 and 10 minutes to deliver. The tests themselves took the students from 15-40 minutes to complete to the best of their ability. Some students were taken out of the classroom and took the test alone or in a small group with a teacher. The teacher assessed the tests and evaluated the students’ written answers, often in collaboration with other teachers who had been involved in testing other students in third grade at the same school (from field notes in 2010 and 2011).

National tests have a dual purpose: to evaluate the student and to evaluate the education (Clarke, Madaus, Horn, & Ramos, 2000; Lundahl, 2009a, 2010). In this way, equity and achievement among students are monitored through the testing of the knowledge of individuals (Lundahl, 2009b; Roth, 2010). This is an example of how evaluations have moved from being supportive tools in education towards forming the grounds for policy directions and steering instruments (Segerholm, 2009). This can be described as governing by objectives and results (Lundahl, 2010). In this way, the schools and teachers are governed by objectives and results at the same time as they govern the students by objectives and results through the tests.

The rising number of Swedish students not meeting the standards in mathematics has been referred to both in national (Skolverket, 2010c) and international evaluations (Skolverket, 2010a). This has become both a concern for the schools and an argument in the debate concerning the quality of education (Lundahl, 2009a; Skolverket, 2010a).

Issues of social justice during Ntm3

Swedish schools are required by law (see the Education Act (Skollag, 2010) and the Discrimination Act (Diskrimineringslag, 2008) to ensure equal access to education and associated activities. Equity is understood here as affording equality to students and can be:
... realized when students have equal access to resources, high-quality teachers, and appropriate instructional support regardless of race, class, gender, and so on. (Parks & King, 2007, p. 406)

Despite the drive for equality, the Swedish school agencies find that the achievement levels of certain groups of students vary markedly. This is seen in students’ final grades in elementary school and in results from national tests, suggesting segregation and a lack of equality in the system at large (Skolinspektionen [Swedish School Inspectorate], 2014; Skolverket, 2012b). Hansson (2011) points out that research has to take on the task of investigating the underlying structures behind pedagogical segregation, where pedagogical segregation is understood as the equivalence of education being affected by the composition of students in the class in regard to different needs, class, gender, background and language. Boys and students with an immigrant background are groups that have been identified as especially disadvantaged in relation to mathematical achievement (Skolverket, 2011a, 2011b), although boys achieved better results in Ntm3 than girls in both 2010 and 2011 (Skolverket, 2011a, 2011b, 2011d, 2011e). The most significant factors for not managing the test in mathematics were parental education level and immigrant origin (Skolverket, 2011a, 2011b, 2011d, 2011e). Boistrup and Norén (2012) state that discourse and social and political conditions in society must be taken into account when it comes to understanding the conditions for achievement of multilingual students. This paper looks in detail at multilingual students, since they are identified as a disadvantaged group, and their situation may throw light on the discursive practice of giving support and its potentially segregating consequences.

**Methods and Selection**

The data for this article are derived from observations and interviews with eight teachers (Table 1) in eight classes at three schools during the first two years of fieldwork (2010-2011) together with the test’s instructions to teachers (Skolverket, 2010b, 2011c). Comments and quotes gathered from the teachers during interviews are supplemented with field notes and observations of talk in the test situations. As data were processed, the results informed the researcher on how to engage in the next step of the research process.
Table 1:
Overview of participating schools, teachers and approximate number of multilingual students. All names are anonymised.

<table>
<thead>
<tr>
<th>Year, location &amp; multilingual students</th>
<th>Teacher responsible for the tests</th>
</tr>
</thead>
<tbody>
<tr>
<td>2010, outer city, &gt; 45%</td>
<td>Maria. 9 years’ teaching experience. Trained as a mathematics teacher.</td>
</tr>
<tr>
<td>2010, inner city, &gt; 35%</td>
<td>Jeanette. 13 years’ teaching experience.</td>
</tr>
<tr>
<td>2010, suburb, &lt; 5%</td>
<td>Ann. 13 years’ teaching experience. Has given national tests before. Trained as a mathematics teacher.</td>
</tr>
<tr>
<td>2010, suburb, &lt; 5%</td>
<td>Margaretha. 9 years’ teaching experience. Has given national tests before. Trained as a mathematics teacher.</td>
</tr>
<tr>
<td>2010, suburb, &lt; 5%</td>
<td>Eva. 12 years’ teaching experience. Has given national tests before.</td>
</tr>
<tr>
<td>2011, outer city, &gt; 45%</td>
<td>Anne. 8 years’ teaching experience. Trained as a mathematics teacher.</td>
</tr>
<tr>
<td>2011, inner city, &gt; 35%</td>
<td>Sara. 21 years’ teaching experience. Trained as a mathematics teacher</td>
</tr>
<tr>
<td>2011, suburb, &lt; 5%</td>
<td>Julia. 12 years’ teaching experience. Has given national tests before.</td>
</tr>
</tbody>
</table>

Methods for collecting and analysing data are theoretically placed within the fields of qualitative research and ethnography (see, for example, Davies, 1989; Madden, 2010). Visual ethnographical methods have been advocated in the production of data (Pink, 2001). Foucault’s theory about discursive formations has been used as a frame of analysis (Foucault, Bjurström, & Torshell, 2011) for identifying and demarcating the discourse on support. Sentences concerning support and students in need were collected from the instructions and teacher talk during interviews and in test situations. Field notes and observations were consulted when recurring themes, issues, statements, concepts etc. were detected. A further review that included switching between video clips of the test situation and teachers’ interviews was then made. Overall connections such as differences and similarities were searched for, as well as issues conspicuously absent either altogether from the data selected for analysis or from certain parts of the data. Relations between findings were explored and foundations in the discourse on support identified. How the teacher was subjected and positioned was discussed through the lenses of the testing, competition and caring discourses found in the test situations in previous analysis of the initial project (Sjöberg, Silfver, & Bagger, 2015).
The discourse on support

Foundations in the discourse on support are presented by displaying overarching tendencies in the talk during interviews and in test situations and text connected to support. The general instructions for the tests informed the teacher about how to approach the student and handle the test. The importance of following these instructions in order for participation in the tests to be equal for all students in Sweden was highlighted in the introduction. In this way, teachers’ actions were directed on a meta-level, showing how the rest of the instructions should be read. These parts are understood as ‘key instructions’. Some of these were the same in both years: The most important thing is that the test material is one (of several) means to evaluate the students’ knowledge. The test shall not be adjusted to such a degree that the goals in the curricula are not tested. If the teacher helps a student beyond this, this must be taken into regard in the evaluation. General instructions are that the materials used can be adapted to the students in a way that the school/teacher finds appropriate (Skolverket, 2010b, 2011c).

The test instructions allow teachers to make necessary adjustments but require them to stay true to the test’s objectives. Teacher talk during interviews and in test situations revealed several examples of occasions when the two requirements of making adjustments and staying true to the objectives came into conflict with each other. These requirements reveal the dual purpose of the test. Since the teacher is also being tested, the urge to stay true to the test might be interpreted as ‘not cheating’; however, the teacher is also the test-giver, who is supposed to test the knowledge of the students. The conflicts that could arise from these opposing positions became apparent in teachers’ stories about providing support, for example, when there was a need to explain the task at hand in the children’s own language to give them access to the mathematical content of the test. Teachers’ ambivalence regarding this can be summarized in two overarching themes: the teacher as a test-taker versus the teacher as a test-giver and the student’s needs versus the test’s needs. This is closely related to two of the main themes underpinning the discourse on support during test situations: issues of equity and teachers’ agency to provide support. These themes and their relation to the subjection and positioning of the teacher will be explained and illustrated in the following.

The teacher as test-giver vs test-taker

The conflict for the teacher between the position of being a test-taker and a test-giver is obvious and is understood as being grounded in the two contravening purposes of the tests. Helping too much could be considered cheating. The
instructions state that assignments can be read aloud and explained, and that it is desirable to help the student to understand the context of the task. Still, the teachers had to interpret this in their practice, and Ann, for example, asked the question: “How much are we allowed to say?” This question could be answered with the guidance of the instructions, as pointed out by Eva: “You are allowed to underline what is said!” In practice, the help that could be provided was limited. Other statements that reveal this limit include Julia’s comment: “You cannot really ask for help, only if you do not understand the question”. Opposing instructions opened up for giving the support needed by stating that it was permitted to help, but that this had to be considered during the evaluation and correction of the student’s test. Supporting actions were sometimes talked about as being forbidden or restricted with reference to instructions or practical circumstances at school outside the teachers’ jurisdiction. I interpreted this to mean that the teacher is aware of the student’s right of equity, but in the position of a test-taker, the teacher does not perceive herself to be able to permit the support needed to secure equity. Equity is here used as something that encompasses students’ development of a sense of efficiency (empowerment) in mathematics together with the desire and capability to learn more about mathematics when the opportunity arises. (Cobb & Hodge, 2007, p. 160)

**Students equity vs the test’s equality**

Further examples of the teachers’ awareness of students’ need for support and right to equity together with the teachers’ experience of not having the agency to provide support were when the teachers of the multilingual students kept identifying language as the thing that needed to be adapted. Maria put it this way: “Did the test constructors think about mathematics and second language learners?” Anne pointed out situations that hindered the support, for example, when she could not translate and the teacher who knew the language was not available: “How could I explain, because there are many who do not understand the language?” Other language issues that were talked about as being a threat to the equity of the student were the fact that the test only exists in Swedish and English. Anne stated with frustration in her voice, “Why can’t the test be in all the major languages: Arabic, Persian, French and Spanish?” Here the teacher talk revealed a kind of alliance against the test writer mentioned by Kasanen and Räty (2007), whereby class teachers stressed loyalty to the students in connection with the national tests. Maria mentioned examples of how the Swedish language hindered the students’ access to the mathematical content: “In an everyday situation it is possible to use peers to, for example, explain the word ‘half’.”
Code switching is when several languages are used to understand and explain something being discussed (see, for example, Norén, 2010). Maria meant that switching between languages would give more students access to the content of the tasks on the test, similarly to how students help each other to understand the everyday mathematics. Maria said: “It can be from both ways, that they can manage this in one language but not in the other”. Providing this level of support during the test was not possible due to the risk of cheating. I interpreted this as the teachers’ focus moving from the support for learning to the controlling of student test-takers. In these situations, the teachers positioned themselves as not having the agency to support students due to the test’s construction, instructions or resources available in school and within the teacher – for example, knowledge of the student’s language.

The test instructions on how to support multilingual students are not always helpful for the teachers, as there is not very much information about the areas that the teachers describe as challenging. The student is of course allowed to use a dictionary, and the teacher is allowed to translate words and to explain tasks. The instructions also say that students who struggle with the language might find the oral test more appropriate than the written version. Both oral and written parts are available in Swedish and English only; there is no mention of providing the students with translations of the tasks into their first language, in order for them to be able to work through the tasks on their own. The instructions do not direct or recommend how schools use their language resources during the tests (Skolverket, 2010b, 2011c). There is a clause stating that the test instructions can be spoken as well as written. But even though teachers are allowed to orally translate the assignments into other languages, the situation remains reminiscent of the “Swedish only” discourse mentioned by Norén (2011, p. 2). Norén (2011) showed how the agency of students and teachers during the test could contribute to a switch of discourse from ‘Swedish only’ to an allowance of using other languages, which in turn could lead to access to the mathematical content and thinking. But this situation could only arise if the teachers had the necessary language skills, which is not the case for all teachers nor all schools. Even when such a resource was available in the form of a teacher who could translate into, for example, Arabic, the class teacher Anne was doubtful. “How much did he help, really? Maybe too much? We do not know. Since they managed those parts ...”. An alternate explanation would have been that the students managed the tests since they got access to the language. I interpreted this as the teacher, due to the two-fold purpose of the test, being subjected and positioned both as a test-giver, who should govern the student and the test, and a test-taker, who should not cheat, due to the two-fold purpose of the test.
Time for competition vs time for caring

A competition discourse is revealed in the teacher talk during interviews and in test situations. Here, a traditional competition is understood as being constituted by all participants following the same rules, performing the same activities, at the same time, with the aim to place individuals in order and thereby subjected as winners or losers depending on their success in some aspect (also see Sjöberg, Silfver, & Bagger, 2015). Schools are compared to each other in statistics based, among other things, on how well their students score in the national tests and are ranked according to these scores in the daily press. Within both the competition discourse and the testing discourse, it is desirable to score as highly as possible. When Maria said: “It is a trial after all, so it should not be incorrect”, she indirectly talks about a measurement of her own teaching and position as a test-taker within the competition and testing discourse.

According to the instructions, the seriousness of the test should be played down in the test situation. This is supposed to help students take the test and is thereby considered a kind of support. But according to the teachers, this kind of talk could have exactly the opposite effect of the one intended. In fact, the teachers talked about the seriousness of the test in order to help the students concentrate for a longer time and withstand feelings of not being able to succeed, confusion and tiredness; “because they are not used to the test situation”. According to Julia, the fact that she talked about the seriousness of the test meant that “… the students were more concentrated”. Julia explained that she tried to instill in the students a feeling of: “This is it!!! And it really is!” The emphasis on achievement and accuracy was also referred to as a help for students to act against their nature: “It is not natural for these young children to do tests”. Instructions about how to help students overcome stress, how to approach students in the test situation or how to talk about the purpose of the test are not apparent in the test guidelines (Skolverket, 2010b, 2011c).

Even if teachers said that the purpose was to find out the level of knowledge in order to know what to teach the student, they also added that it was quite obvious that the test was also about something else: to win, to not make mistakes and not fail the test. These objectives seem to concern all test-takers in the classroom, both teachers and students, since they are all being evaluated. This pressure is similar to what Reay and Wiliam (1999) found when the schools’ existence is threatened by poor test results, namely that:

… individual teachers are under increasing pressure to improve the scores achieved by the students, irrespective of the consequence for students’ achievement in wider terms. (ibid., p. 352)
After the teacher had collected the completed test papers, the time for competition seemed to fade away. The caring discourse then emerged and it was possible for the teachers to position themselves within this discourse. This is, for example, what happened when teachers said that they talked to students individually about tasks in the test that they had failed. All teachers talked about this and how they helped the students to understand the tasks and gave them a second chance to answer. These events are interpreted as: when the time for competition has passed, help and adjustments to meet the students’ needs are allowed. In other words, there were circumstances when the rules regarding the test’s equality hindered parts of the testing of individual students’ knowledge, and testing instead took place after test hours. The teachers could then act within the testing and caring discourse and could position themselves more as caretakers and test-givers, rather than as test-takers. Apparently it was easier to consider the students’ right to equity outside test hours, whilst during the test it was the equality of the test that was considered. However, in some cases, the teacher is still positioned as a test-taker in connection with the evaluation of the test. For example, on one occasion a teacher asked: “How do you correct a torn-up test?” The underlying assumption being that searching for the score is the important thing when evaluating a torn-up test, rather than evaluating what has preceded the action of destroying the test as well as the kind of needs the student might have.

Conclusions

This article has three main conclusions. First, the testing discourse positions the teacher as both a test-giver and a test-taker. This, I suggest, is likely connected to the dual purpose of the test of both monitoring the students’ knowledge and the quality of education. Second, these positions affect the support that the teacher feels that it is right to provide and bring into conflict the need for equality of the test and the need for equity for the student. And third, the need for equality shifts the focus in the test situation from learning to the governing and controlling of individuals.

The discourse on support during the tests is emerging in a field of tensions between other prevalent discourses. Discourses govern the positions teachers can assume when providing support and thereby affect how the support may turn out in practice. Since the teacher is torn between the position of a test-giver and a test-taker, the actual test situation can sometimes hinder support during the test. When teachers followed the instruction to make appropriate adaptations so that the student could access the test, they did it mainly within the caring and testing discourse and outside of test hours. During test hours, teachers were more likely
to be positioned as test-takers within a discourse of competition. The correction of tests means evaluating the school’s teaching and thus has implications for local, national and international statistics. Time taken for correction is therefore a kind of test hour for the teacher. Re-correction of tests is conducted to look into whether teachers are grading tests properly. Teachers’ legitimacy could be questioned if tests are not properly handled, which might be part of the explanation for why the teacher in this article assumed that the torn-up test should be corrected.

Underlying the discourse on support are issues of equity and teachers’ agency to give support. These are like the weights on a scale – constantly affecting each other. The solution might not necessarily be to separate the two, eliminating either from the balancing act (the national tests), but to be aware of the inherent conflict in order to balance them so that neither takes over at the expense of the other. If issues of students’ equity take precedence, the test cannot assess the teacher’s practice. Meanwhile, if the teacher is positioned in the role of a test-taker too forcefully, he or she will not experience agency to give support, which leads to a lack of equity for the students. Teacher talk revealed experiences of lacking the agency to offer support, especially for multilingual students. This might affect the students’ accessibility to the mathematical content and, if so, their achievement in the tests. When teachers act like this, they are positioned within the testing discourse as test-takers giving a test. There is a risk that the situation of testing contributes to pedagogical segregation which – later on – can be seen in statistics and, additionally, affects students in need of support in the test situation. There are questions to be raised regarding the focus on the teachers’ responsibility, in order to make sure that appropriate adjustments of the tests are made and that necessary resources are available as instructions assume. What the term appropriate means in the instructions is not clear and might come into conflict with the teachers’ task to give support when positioned as test-takers. What are the conditions for appropriateness, who decides and what is the aim of enacting this appropriateness? Teachers tend to a larger extent to be given or exercise agency in situations where they act within the caring discourse. If the test instructions took this part of the teachers’ role into account, it might help teachers to secure students’ equity as test-takers.

A further quest for research and practice is to find out how to support the student in the test situation without interfering with the measurement of the quality of mathematics education provided, or vice versa. There is a need for further investigation of how test results and the support provided to students are influenced by the dual purpose of the test and the educational dilemma between considering the student’s equity and the test’s equality. In addition, how and
whether these aspects are connected should be further investigated. Must the equity of the student and the equality of the test be weighed against each other? If these issues are not handled in practice, there is a risk that the dual purpose of the test will hinder supportive actions, which in the long run threatens both purposes.

Notes
1 The project “What does testing do to pupils?”, conducted together with Gunnar Sjöberg, Eva Silfver and Mikaela Nyroos, is financed by the Swedish Research Council.
2 The Swedish National Agency for Education’s online information system on results and quality, http://siris.skolverket.se/siris/?p=Siris:1:0.
3 The students were asked to work together to solve mathematical problems while the teacher observed their interaction in order to assess the mathematical skills of each student.
4 For example, when instructions told teachers to play down the seriousness of the test situation to calm the students while the teachers themselves talked about the seriousness of the test to motivate the students to do their best, or the fact that neither the instructions nor the teachers talked about learning.
5 i.e. the everyday context wherein the mathematical task is embedded.

References


Svensk sammanfattning

**Elevens eller provets likvärdighet? - Stöd i samband med det nationella provet i matematik i det tredje skolåret**


Nykterord: nationella prov, matematik, tredje klass, särskilda behov, särskilt stöd.
Comparison of Two Test Approaches for Detecting Mathematical Difficulties

By Pernille Bødtker Sunde & Pernille Pind

Abstract

According to Encyclopedia on Early Childhood Development, early detection of constrained arithmetic comprehension is important to prevent development of severe mathematical difficulties (Jordan, 2010). For that purpose, efficient diagnostic tools are suitable. In this paper, we evaluate two tests with different approaches to diagnosing mathematical difficulties: a norm-referenced test measuring general mathematical achievement (MAT) and a criterion-referenced test measuring comprehension and fundamental insight in arithmetic principles (RoS/test). We compare the outcomes of the two tests when used on 59 children in 2nd and 3rd grade, and discuss similarities and discrepancies in relation to the underlying evaluation principles for diagnosing difficulties in mathematics.

Keywords: diagnostic test, norm-referenced, criterion-referenced, strategies in arithmetic, mathematical difficulties, assessment interviews.

Introduction

Detecting difficulties with mathematics at an early stage is, according to Jordan (2010), important in order to take action to prevent mathematical difficulties later
in life. If students’ fundamental mathematical difficulties remain undiscovered, unsatisfactory calculation routines will develop into well-established habits (Haseler, 2008). Moreover, comprehension difficulties on a basic arithmetic level may block learning of more advanced mathematics later in life (Haseler, 2008; Jordan, 2010).

Students’ mathematics performance at the beginning of school predicts later achievements (e.g. Desoete & Grégoire, 2006; Geary, Hoard, Nugent, & Bailey, 2013). Typically, students are identified with mathematics difficulties within the key areas of number sense, number knowledge, and strategies in arithmetic as early as the first years of school or before. Later on, however, the same students can display more complex difficulties (Desoete & Grégoire, 2006; Gervasoni, 2005). Research within neuroscience and mathematical cognition documents the relationship between, on the one hand, number sense, number knowledge, and strategies in arithmetic, and, on the other hand, development of formal mathematics (Butterworth, Varma, & Laurillard, 2011; Chu, vanMarle, & Geary, 2015; Feigenson, Libertus, & Halberda, 2013; Geary & Brown, 1991; Ostad, 2010). Diagnostic tests aimed at detecting early mathematical difficulties should, therefore, assess number sense, number knowledge, and strategies in arithmetic with as high sensitivity as possible in order to identify students with problems, or potential problems, with mathematical comprehension.

When testing for (potential) mathematical difficulties, two questions arise: 1) does the child have difficulties or not; and, 2) what action should be taken regarding the difficulties. To answer the first question, the test should actually be able to detect mathematical difficulties. To answer the second question, the test should provide information on the nature of the difficulties: for example, the child’s level of comprehension within a given topic. According to Popham (2009, p. 90) “… a diagnostic test helps identify a student’s learning problems so teachers can provide instruction to remedy those problems”.

In this paper, we will address some principles for assessment tools aimed at diagnosing difficulties in mathematics, and discuss their pros and cons when applied to real cases. As an example, we present and compare the results from two different diagnostic tests applied to 59 children.

Assessment tools or tests can be described with respect to many aspects. Here, we will focus on three: 1) norm-referenced vs. criterion-referenced tests; 2) whether the tests sample a broad range or specific elements of arithmetical and mathematical topics; and, 3) the interpretation of diagnostic results in relation to subsequent action.
Norm-referenced or criterion–referenced tests

Norm-referenced assessment ranks students according to relative measurements in relation to a population mean (Popham & Husek, 1969; Taylor, 1994). Norm-referenced tests are designed to compare individuals or organizations, based on the assumption that learning can be objectively compared between individuals. Furthermore, learning is considered a continuous linear process; that is, learning is acquired gradually (Taylor, 1994), since continuous variables will result in “bell-shaped” performance curves (Mazzocco, 2005) – just like measuring the height of the students.

Criterion-referenced assessment determines if a student has achieved a certain learning objective as described, for example, in public educational standards (Popham & Husek, 1969; Taylor, 1994). Criterion-referenced tests are devised to make decisions about individuals as well as treatments: for example, instructional programs (Popham & Husek, 1969). Criterion-referenced testing is based on the assumption that learning is an individually varying process, and that documenting the students’ development can help the teachers to focus their teaching and change teaching practice towards “what works” (Taylor, 1994). While norm-referenced tests can be said to assess skills in a summative manner (what have children learned or not learned relative to the population mean), criterion-referenced tests are formative assessments of the learning process (where is the student in relation to the learning objective).

Diagnostic thresholds are defined very differently for norm-referenced and criterion-referenced tests. For norm-referenced tests, the cut-off values are defined as the lower percentiles of the population distribution (ordered to have the shape of a normal distribution). It is common to use a score lower than the 20th to 25th percentile to diagnose difficulties in mathematics (Fletcher, Francis, Morris, & Lyon, 2005; Geary, 2004); that is, the lowest performing 20 or 25% of the student population from which the distribution has been created are considered as having learning difficulties. For criterion-referenced tests, the diagnostic threshold is defined objectively in relation to a specific learning objective and is ‘on-off’ in nature: either the student has reached the desired criterion or not (Popham & Husek, 1969).

The pros and cons of using norm-referenced or criterion-referenced testing have long been discussed (Lok, McNaught, & Young, 2015; Taylor, 1994; van den Heuvel-Panhuizen & Becker, 2003). The aim in this paper is to discuss their implications for diagnosing mathematical difficulties, and the opportunities for interpretation of results with respect to subsequent action.
Range of assessed topics

Children with mathematical difficulties often show severe deficits within specific arithmetical and mathematical topics while performing well in others. For a standardized achievement test, assessing a wide range of topics, the result averages across items; therefore, children can perform within the average range despite their difficulties. That is, difficulties in one area can be compensated by achievements in other areas (Geary, 2004; Mazzocco, 2005).

Even when a test result indicates difficulties, the range of topics assessed is important in relation to the interpretation and action on results (Mazzocco, 2005). According to Geary (2004), tests including a wide variety of problem types provide limited information about the type of problem a given student may have. To identify children with mathematical difficulties, further tests are needed to supplement standardized achievement tests.

Interpretation and action on diagnostic results

When performing assessments or screenings, the objective of the assessment should correspond with the characteristics of the tool, since that is decisive for the outcome knowledge and possibilities for reflection, and hence capacity to act on the accomplished knowledge (Black & Wiliam, 1998; van den Heuvel-Panhuizen & Becker, 2003). Diagnostics should be followed by action (Fletcher et al., 2005; Popham, 2009). That implies that assessment should provide relevant information on students’ achievement: information that provides direct clues for making didactical decisions (e.g., van den Heuvel-Panhuizen & Becker, 2003).

Popham (2009) outlines four parameters that legitimate diagnostic tests should fulfil:

*To be truly diagnostic, such tests need to (1) measure a modest number of significant, high-priority cognitive skills or bodies of knowledge; (2) include enough items for each assessed attribute to give teachers a reasonably accurate fix on a test taker’s mastery of that attribute; (3) describe with clarity what the test is assessing; and (4) not be too complicated or time-consuming. (Popham, 2009, p. 91)*

Two tests exemplifying different test approaches

In the following, we present two examples of different approaches with regard to diagnosing children in difficulties or at risk of developing difficulties: a norm-referenced and a criterion-referenced test. The tests differ in the range of topics they assess and on the information on test result (number of errors within a topic vs. comprehension level of answers).
Firstly, we describe the two tests. Secondly, we present and compare results of the two tests applied on 59 Danish children in 2nd and 3rd grade (in Denmark, children in 2nd and 3rd grade are usually 8-9 years old). Lastly, we discuss the different approaches to diagnostic testing represented by the two tests with respect to their capability to detect children with (potential) difficulties, and the possibilities to interpret the test outcome with respect to focused intervention.

**Descriptions of the tests**

**The norm-referenced test: MAT**

MAT is a standardized norm-referenced test (Jensen & Jørgensen, 2007) commonly used for diagnostic testing in Danish primary schools. MAT tests a wide range of mathematical skills within number and algebra, geometry, and applied mathematics. The test is conducted as a pen and paper test for a whole class, adapted to the curriculum for each grade (separate tests for each grade). In the present analysis, we used MAT1 for 2nd grade and MAT2 for 3rd grade (hereafter referred to as ‘MAT’). The test consists of simple calculation tasks, tasks on number knowledge and the number system, problems on geometric shapes, measuring length and area, problems on statistics and probability, etc. The number of items within each topic varies from one to more than ten. The test is not time restricted and time is not monitored. The test procedure normally takes 2-3 hours in grades 1-3, 3-4 hours in grades 4-5 and 4-6 hours in grades 6-9. Calculation of test scores takes 2-4 hours per class. If a child is diagnosed with learning difficulties (Table 1), follow-up testing is recommended to identify the specific characteristics of the difficulties.

The test score (total number of errors) is transformed to a proficiency level represented by a C-value: a scaling of test scores comparable to a Stanine-scale. For each student, the number of errors is summed and the results are scored on an ordered metric scale relative to a population distribution based on 1800 students from 1st to 9th grade (i.e., 200 per grade). The C-values are obtained by dividing the theoretical normal distribution into 11 intervals (C0-C10), each of which has a width of 0.5 standard deviations, excluding the first and last, which are just the remainder (the tails of the distribution). The mean lies at the centre of the fifth interval (C5). Error norms and corresponding C-values are provided for each grade. The percentage of children in each group (based on the ordered metric scale and theoretical normal distribution) and the corresponding diagnostic characteristics as outlined by Jensen and Jørgensen (2007) are shown in Table 1.
Table 1:

Description of MAT diagnostic C-scores: Percentage of children of the Danish norm population in each C-category, corresponding diagnostic characteristics as outlined in the test manual (Jensen & Jørgensen, 2007) and the diagnostic groups used for comparison with the RoS/test (our grouping).

<table>
<thead>
<tr>
<th>C</th>
<th>% of children</th>
<th>Diagnostic characteristics</th>
<th>Diagnostic groups for comparing with RoS/test</th>
</tr>
</thead>
<tbody>
<tr>
<td>C0-C1</td>
<td>4</td>
<td>Severe learning difficulties</td>
<td>Severe learning difficulties</td>
</tr>
<tr>
<td>C2</td>
<td>7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C3</td>
<td>12</td>
<td>Learning difficulties</td>
<td>Below average</td>
</tr>
<tr>
<td>C4</td>
<td>17</td>
<td>Below average</td>
<td></td>
</tr>
<tr>
<td>C5</td>
<td>20</td>
<td>Average</td>
<td></td>
</tr>
<tr>
<td>C6</td>
<td>17</td>
<td>Above average</td>
<td>Average or above average</td>
</tr>
<tr>
<td>C7</td>
<td>12</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C8</td>
<td>7</td>
<td>Considerably above average</td>
<td></td>
</tr>
<tr>
<td>C9-10</td>
<td>4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The criterion-referenced test: RoS/test

RoS/test (Sunde & Pind, 2014) is a criterion-referenced test assessing the qualitative use of strategies in mental addition (comprehension in arithmetic ability) inspired by Ostad (1997; 2008). The test focuses on mental strategies for solving basic arithmetic problems as a key assessment criterion since the correlation with acquisition of formal and more advanced mathematics is well documented (Geary & Brown, 1991; Ostad, 2010).

The test is conducted as an interview between teacher and child (qualitative assessment), where the child is presented with flashcards picturing addition combinations while the teacher scores the child’s use of strategy (quantitative assessment). The flashcards consist of 36 single-digit addition combinations with numbers from 2 to 9 including all ties – for example, 4 + 4 – since they are important sums in the strategy decomposition. The time used is 10-15 minutes per child, including data processing and interpretation.

The task specific strategies used for solving mathematical problems are categorized in backup-strategies (BC) and retrieval strategies (RT). BC-strategies do not depend on comprehension skills, as the child follows a fixed plan like counting fingers. When using RT-strategies, the child retrieves information stored in the mind. This is the result of storing and cognitive processes based on comprehension, not rote learning (Ashcraft, 1992; Ostad, 2010). In RoS/test, the
decomposition strategy, where the sum is derived by transforming the task into known sums, is categorized as a retrieval strategy, since it involves the use of memorized number facts through direct retrieval. That is consistent with Ostad’s (1997; 2008) use of the term. Decomposition is considered more advanced than direct retrieval since using such strategies demands the understanding of decomposition of numbers.

The RT-strategy decomposition is further subdivided in two strategies: decomposition with addition and decomposition with subtraction, referring to whether the intermediate sum is lower or higher than the final result and thus demanding subsequent addition or subtraction (e.g., $4 + 5 = 4 + 4 + 1$ or $5 + 5 - 1$). Furthermore, the test operates with two error categories: 1) the child gives up and 2) the child miscalculates (see Table 2).

Table 2:
Strategy categories of RoS/test (Sunde & Pind, 2014) compared to the strategies used by Ostad (1997; 2008).

<table>
<thead>
<tr>
<th>Strategy</th>
<th>BC or RT</th>
<th>Description The child:</th>
<th>Ostad (1997; 2008)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Gives up</td>
<td></td>
<td>has no strategy and gives up</td>
<td>-</td>
</tr>
<tr>
<td>2 Miscalculate</td>
<td></td>
<td>miscalculates without noticing</td>
<td>-</td>
</tr>
<tr>
<td>3 Counting all</td>
<td>BC</td>
<td>counts both addends and then all</td>
<td>A1a, A1b</td>
</tr>
<tr>
<td>4 Counting on</td>
<td>BC</td>
<td>counts on from one addend</td>
<td>A1c, A1d, A1e, A1f, A1g, A1h</td>
</tr>
<tr>
<td>5 Direct retrieval</td>
<td>RT</td>
<td>knows the answer</td>
<td>A2a</td>
</tr>
<tr>
<td>6 Decomposition +</td>
<td>RT</td>
<td>decomposes the addends and uses addition (e.g. $4 + 5 = 4 + 4 + 1$)</td>
<td>A2b, A2c</td>
</tr>
<tr>
<td>7 Decomposition -</td>
<td>RT</td>
<td>decomposes the addends and uses subtraction (e.g. $4 + 5 = 5 + 5 - 1$)</td>
<td>-</td>
</tr>
</tbody>
</table>

Note: BC: backup strategy, RT: retrieval strategy.
On the basis of a combination of grade-specific threshold values of the proportionate use of the different strategies (criteria), the child is categorized into one of three guiding diagnostic groups of mathematical comprehension relative to age: 1) in or at severe risk of developing difficulties, intervention required; 2) at risk but in conceptual progress, attention recommended; and, 3) in adequate progress. The test is the same for all grades but threshold values for achievement of the learning objective differs among grades.

In addition to the quantitative assessment, the interview provides the teacher with a qualitative insight into their students’ comprehension of mental strategies in addition and the underlying number understanding.

Application of different diagnostic approaches on a student sample

Methods

We applied both tests in the middle of the school year to 59 children from one 2nd grade class (19 children) and two 3rd grade classes (40 children) from a Danish medium sized (340 students) countryside public school with a culturally and ethnically homogeneous student population (all had Danish as their mother tongue).

To compare the diagnostic outcomes of MAT and Ros/test, we grouped the initial 11 MAT C-scores (Jensen & Jørgensen, 2007) into three diagnostic groups, corresponding with the diagnostic groups in RoS/test (see Table 1): 1) C0-C2 severe learning difficulties comprising the 11% of the Danish norm population with most errors; 2) C3-C4 below average encompassing the following 29% of the norm population having next most errors; and, 3) C5-C10 the remaining 60% of the children having least errors and thus representing the average or above average fraction of the norm population. The lowest performance category (C0-C2: calling for immediate intervention) encompasses a similar or slightly lower population fraction than used in other studies of diagnosing comprehension problems (e.g., 20-25%) (Fletcher et al., 2005; Geary, 2004); the second group (C3-C4), however, represent a slightly larger fraction of the norm population of students performing below average compared to other studies (Geary, 2004). We argue that this group should also incorporate the children below average since it could be assumed that potential later difficulties will arise from this group.

The consistency of the diagnostic outcomes of MAT (C-values) and RoS/test (diagnostic group 1-3) were assessed on the basis of the correlation (Spearman Rank Correlation Coefficient) of test scores of the two tests, and as the proportion of children diagnosed similarly (same diagnostic group: 1, 2 or 3), inconsistently
(deviating by 1 score), or oppositely (classified at 1 in one test and 3 in the other test and vice versa).

The relative difference in the number of students classified as requiring intervention (positives) by one test out of those classified as in adequate progress (negatives) by the other test was tested with Fisher’s exact test.

As a post hoc analysis, examples of test outcomes for children diagnosed as having severe learning difficulties for both tests are given, and the characteristics of those children that were classified totally differently by the two tests were subsequently analysed in detail.

Results

Appearance, correlations and comparisons of diagnostic outcome

Of the 59 children, the MAT test classified 6 (10%) as having severe learning difficulties, 22 (37%) as having below average skills, and 31 (53%) as having average or above average skills. The RoS/test classified 14 (24%) as having clearly unsatisfactory strategies, 5 (8%) as having somewhat unsatisfactory strategies, and 40 (68%) as having satisfactory strategies for their age.

There was only a modest correlation between the scores of students’ mathematical capability in the norm-referenced test, MAT, and the criterion-referenced test, RoS/test ($r_s = 0.297, p = 0.022, N = 59$).

In terms of agreement with respect to diagnostic groups (1: intervention required; 2: attention recommended; 3: adequate progress), the two tests categorized 29 (49%) of the children similarly, 25 (42%) in neighbouring groups, and 5 (8%) in completely opposite groups (Table 3). All five cases of complete diagnostic mismatch (four 2nd grade students and one 3rd grade student) were scored as requiring intervention by RoS/test and average or above average by the MAT test. This means that one sixth (5 of 31) of the students in the highest achieving diagnostic group by MAT were categorized as having inadequate strategies and scored as requiring intervention by RoS/test. In comparison, none of the 40 children categorised as being in adequate progress by RoS/test were categorised as having severe learning difficulties by MAT (Fisher’s exact test: $p = 0.013$). This implies that RoS/test is more likely to diagnose children scored as in adequate progress by MAT, as requiring intervention, than the opposite (MAT scoring intervention needed when RoS/test scored adequate progress).
Table 3:
Matrix of RoS-groups and MAT C-values.

<table>
<thead>
<tr>
<th>Group</th>
<th>RoS</th>
<th>MAT C-scores</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>C0</td>
<td>C1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
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Note: The MAT C-values have been pooled in 3 groups corresponding with the RoS-groups: 1) in or at severe risk of developing difficulties and intervention required, 2) at risk but in conceptual progress, attention recommended, and 3) in adequate progress. White areas show correspondence of RoS/test and MAT in diagnosing. Light grey areas show moderate discrepancy while dark grey areas show considerable discrepancy between the two tests.

Of the 25 cases of moderate discrepancy between main categories suggested by the two tests, the MAT test provided a lower score than the RoS/test in 18 cases, and a higher score in 7 cases (Table 3).

Post hoc descriptions of test results for students diagnosed in severe learning difficulties

The general test results for students scoring C0-C3 in MAT showed an abundance of errors within the topic number and arithmetic – especially within subtraction and multiplication (division is not assessed) – compared to their peers. When investigating the responses to individual test items with wrong answers, it appeared that the students in some cases did addition instead of subtraction, misunderstood the task, made simple calculation errors, or did not answer. All error categories count as errors in the overall diagnostic category. For errors within geometry (recognition of geometric shapes by counting circles, rectangles, and triangles) and patterns, the cause of errors could be miscounting, misunderstanding of the task, or other explanations. The number of errors within these topics equalled that of their peers.

Students scored as requiring intervention in RoS/test used counting on more than 70% of the test items and a varying proportion of direct retrieval strategies, miscalculations, and giving up. It was primarily on the items where the sum equals 10 or on ties (e.g., 3 + 3) that the strategy direct retrieval was used. If decomposition was used at all, it was almost entirely decomposition with addition and on near-ties (ties + 1, e.g., 3 + 4), or on sums with +9 (e.g., 9 + 5).
Descriptions of cases of total mismatch in the diagnosing of MAT and RoS/test

A 3rd grade student scored C5 in the MAT test, but her use of strategies was scored as requiring intervention according to the RoS/test (counting all: 3%; counting on: 64%; and, direct retrieval: 31%). During the assessment interview in the RoS/test, she described how she tried to visualise the numbers in her mind and make them melt together to provide the answer: “... but it doesn’t work”. Instead, she counted objects. With numbers greater than the number of fingers on her hands, she used objects in the classroom or imagined her classmates and counted them. When performing the MAT test, she used almost three hours compared to her classmates using 1½ to 2½ hours, and the paper was covered with counting marks. In this way, she managed to solve most of the calculation tasks correctly. An investigation of the errors in MAT revealed that most of the errors were within the topic number and arithmetic.

A student scoring C7 in MAT only used two strategies in 2nd grade: counting on (64%) and direct retrieval (36%). She did not show any signs of linking sums, in that she did not use the fact that she knew, for example, 2 + 2 = 4 to reason that 2 + 3 = 5 by decomposing to 2 + 2 + 1. Instead, she chose to count on from 3. Her few errors in MAT showed no overall pattern.

A 2nd grade student scoring C10 in MAT had an even more inadequate strategy use. She miscalculated in 6%, counted on in 72%, and used direct retrieval in 22% of the cases. She had very few errors in MAT, and hence achieved category C10.

The strategy use for the last two students in 2nd grade, scoring C5 in MAT, showed the same overall pattern regarding strategy use in the RoS/test and an abundance of the errors in MAT within the topic number and arithmetic.

Discussion

The capability to detect children with mathematical difficulties

Scores obtained in the norm-referenced test, MAT, and the criterion-referenced test, RoS/test, were only modestly correlated, and one out of six students scored as having adequate skills in MAT was diagnosed as in need of intervention by RoS/test. This indicates that the tests assessed partly different aspects of mathematical capability. A likely reason for this discrepancy is that MAT assesses a wide range of mathematical skills (ranging from number knowledge to geometry and applied mathematics) while RoS/test focuses narrowly on the comprehension processes.
MAT only assesses the child’s ability to derive the right answer (skills) and all the test items in MAT1 (applied to the 2nd grade students) on, for example, arithmetic are within the counting area (numbers and results below 20). The possibility of completing the arithmetic items in the MAT test correctly only by finger counting probably blurred the children’s underlying difficulties of arithmetic comprehension. Furthermore, because MAT is assessing a wide range of topics, the ‘test signal’ of inefficient understanding within arithmetic may be diluted by high or adequate performance within the other topic areas (Geary, 2004; Mazzocco, 2005). A thorough investigation of the MAT test for the students scoring C5 in MAT showed that this was indeed the case, as most of the errors were within the topic number and algebra.

The cases where the two tests disagree are of special interest, and special attention should be on the cases with a total mismatch. If a child is wrongly diagnosed as in adequate progress (false negative), there is a risk that difficulties will be unattended until they become so obvious that intervention is difficult. In our investigation, we only saw cases where MAT scored in adequate progress and the RoS/test scored the same student in intervention needed (Table 3). The detailed investigations of the five cases of total mismatch showed that, in these cases, some degree of comprehension constraint is obvious in the RoS/test results. The students’ strategy uses are highly unsophisticated for students in 2nd and 3rd grade, as they all used a high proportion of counting.

The strategy use described in the examples is an indication of poorly developed number sense, which is an important factor in mathematical difficulties (Feigenson et al., 2013; Wilson & Dehaene, 2007). Ostad (1997; 2010) found that children without difficulties are characterized by having many adequate strategies and a progression in the development throughout school towards RT-strategies. Children using RT-strategies at an early age (6-7 years) are less likely to develop difficulties in mathematics. On the contrary, children with difficulties have few and inadequate strategies, primarily BC-strategies, and they fix their choice of strategies early in school with little subsequent progression (ibid.).

The result of Fisher’s exact test implies that the focused criterion-referenced approach exemplified by the RoS/test is less likely to diagnose false negatives. This approach could thus be a relevant choice for an evaluator who wishes to safeguard with respect to missing as few students with real comprehension problems as possible.
Information provided by the test outcome in relation to focused intervention

As the descriptions of the test results imply, interpretation of a test based on right-or-wrong answers demands a high degree of indirect interpretation, or guessing, about what and how the children were thinking when answering. Some wrong answers are due to comprehensional constraints and others due to misunderstandings; therefore, insight into, for example, the children’s number understanding and arithmetical comprehension is difficult to obtain.

When a wide range of topics are assessed, it complicates decisions on action: where to start and what to do. To initiate focused intervention for a child diagnosed in learning difficulties in this case demands a great deal of detective work and further testing (Geary, 2004; Mazzocco, 2005). This could lead to unfocused initiatives.

A test assessing a single topic will only diagnose difficulties within that specific topic. Difficulties within other topics will not be detected. It is, therefore, of great importance that the topics chosen for diagnostic assessments are key topics in relation to mathematical difficulties. The strength of assessing few topics is that the action can be focused and it is less complicated to plan individually adapted interventions.

Different test methods provide differences in the qualitative insight into students’ levels of comprehension (Fletcher et al., 2005; van den Heuvel-Panhuizen & Becker, 2003). Structured teacher-student assessment interviews provide detailed knowledge on the student’s learning and comprehension. It thereby provides a room of reflection for the student as well as the teacher; moreover, the teacher is given direct insight into the action needed to improve the learning outcome of the student (Clarke, Mitchell, & Roche, 2005; Desoete & Grégoire, 2006) and, furthermore, it has the potential to change teachers’ mind-sets, prompting them to make changes in teaching practice (Clarke & Hollingsworth, 2002; McDonough & Clarke, 2005).

Summary and conclusion

When diagnosing mathematical difficulties or risk of developing difficulties, the assessment tools should be able to actually detect these cases, and, if possible, inform the evaluator about the specific type of comprehension problem. From this, it follows that the occurrence of false positives (warning of a non-existing problem) are more acceptable than false negatives (ignorance of a real problem). A false positive in the worst case only costs unnecessary resources for unneeded intervention, and under proficient testing routines they will soon be identified as
false alarms. An undetected problem of mathematical comprehension, however, may undermine years of succeeding teaching effort with potential lifelong consequences for the student. We have presented results that indicate that assessing a broad range of subjects in a norm-referenced approach has the risk of missing at least some of the potential cases of mathematical difficulties.

An important part of diagnosing is the subsequent action. The criterion-referenced approach is linked to learning objectives and could, therefore, provide clues for what to do in the classroom. The direct interaction with the student during testing, for example, by interviewing, gives the teacher immediate and direct insight in the student’s level of mathematical comprehension.

Comparing a norm-referenced and a criterion-referenced approach highlights the importance of using an appropriate assessment tool when screening for difficulties. The assessment design should be considered carefully before using and interpreting a test.

If the aim is to detect difficulties with mathematics, we argue that, at least early on in school, criterion-referenced tests focussing on a few key factors of comprehension in mathematics provide more information about the students’ learning problems and specific needs for special intervention than norm-referenced tests, assessing a wide range of mathematical skills.

Acknowledgement

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References


Dansk resumé

En sammenligning af to testprincipper til diagnosticering af matematikvanskeligheder


Nøgleord: diagnostisk test, norm-baseret, kriterie-baseret, hovedregningsstrategier, matematikvanskeligheder, samtaletest.
Specific Training of Working Memory and Counting Skills in Kindergarten

By Kaisa Kanerva & Minna Kyttälä

Abstract

Early counting skills at preschool age rely on children’s working memory resources and predict children’s later achievement in mathematics at school. The aim of this study was to study and compare the effects of four different working memory training tasks in improving working memory (near transfer) and early counting skills (far transfer) in preschool children. The training conditions were verbal and visuospatial short-term memory training and verbal and visuospatial working memory training. Ninety-nine children were randomly assigned to one of four working memory training groups or to an active or passive control group. The computerized training was conducted twice a week for a five-week period in kindergarten settings. Our results show that specific working memory training did not enhance performance in working memory tasks or performance in tests of early counting skills. Our study suggests that a deeper understanding of the utility of different working memory training methods requires further studies that focus on the boundaries of the amount of training needed to enhance children’s working memory and counting in kindergarten settings.

Keywords: working memory training, counting, children, working memory subcomponents.
Introduction

Working memory (WM) refers to the ability to concurrently maintain and manipulate information over short periods of time (Baddeley, 1986, 2000). WM is involved in higher-order cognitive tasks, such as learning, reasoning, fluid intelligence, and problem solving (Cowan, 2014; Engle, 1992; Geary, Hoard, Byrd-Craven, Nugent, & Numtee, 2007; Preßler, Krajewski, & Hasselhorn, 2013). For example, in counting backward from five to one, the child has to maintain the digits in his or her memory while monitoring the phase of counting and finding the next digit from the mental number line. WM thus has an important role in active regulation of relevant information to accomplish complex cognitive tasks such as counting. In this study we investigate the possibility of enhancing six-year-old children’s counting skills by training WM in kindergarten settings.

Early counting and working memory

In preschool years, early counting skills and number sense develop, starting with subitizing and continuing to verbal counting and understanding digits by placing them on a mental number line (Von Aster & Shalev, 2007). Counting consists of the ability to use the number words in counting forward and backward, counting while pointing at objects, and understanding and applying the cardinal principle of counting (Van de Rijt & Van Luit, 1999). Before the start of formal school, the numerical abilities needed are thus verbal counting and using counting to determine exact quantities (Gallistel & Gelman, 1992).

Children’s early numerical skills and counting are related to WM (Bull & Scerif, 2001; Bull, Espy, & Wiebe, 2008; Noël, 2009) along with domain-specific number skills (Hornung, Schilz, Brunner, & Martin, 2014). The WM consists of a passive storage system (short-term memory, STM) and the active concurrent processing functions (working memory, WM); these immediate memory systems are needed in early mathematics (Noël, 2009; Ranghubar, Barnes, & Hecht, 2010). Separate and independent WM subcomponents are specialized in maintaining verbal or visuospatial information for short periods of time and in coordinating and integrating the information in memory (Baddeley, 1986, 2000). The phonological loop is responsible for the temporary storage of verbal information, for example, storing a string of digits in memory for a short period of time. The visuospatial sketchpad is in turn responsible for the storage of visuospatial information, for example, mental images of digits or manipulation of the mental number line. The central executive coordinates the maintaining and manipulating of information in working memory, for example, in backward counting, the direction and phase
of counting. The episodic buffer integrates information from subcomponents and from long-term memory in the working memory.

It has been suggested that counting especially requires the resources of the phonological loop and the coordination processes of the central executive (Noël, 2009). However, the contribution of different WM components in counting in preschool aged children and more complex mathematical skills later at school is not clear, and is affected by a child’s cognitive development, task-specific skills, and the requirements of the mathematical tasks and strategies (Ranghubar et al., 2010).

WM develops extensively in childhood. Development in several areas of cognitive systems, including an increase in the processing speed, attentional processes, and strategies, underlie the increase in WM capacity (Cowan & Alloway, 1997). Along with these, during the preschool and school years, the emphasis of different WM components in cognitive processing changes. Young children, under about seven, seem to rely more on visuospatial WM resources in cognitive tasks, whereas older children start to rely additionally on verbal WM when processing visual or phonological material (McKenzie, Bull, & Gray, 2003). Since WM components contribute differently to early counting, it is justified to examine their causal role by training them separately and studying the effects of training on counting.

The effectiveness of working memory training

The objective of WM training is to ensure transfer to other tasks that require similar resources (Klingberg, 2010). Transfer refers to the effect the practicing of one task has on the performance of another, different task (Barnett & Ceci, 2002). If the transfer is evident between two structurally similar tasks (e.g., two working memory span tasks with different items), it is called near transfer. If the transfer is evident between two structurally different tasks (e.g., a working memory span task and a fluid intelligence test), it is called far transfer. Consequently, the goal has been to train general WM mechanisms and show a transfer to higher level cognitive tasks, for example, mathematics or fluid intelligence.

Adaptive WM training has been suggested to have an effect on children’s performance in a wide variety of non-trained tasks in different age groups (Klingberg, 2010). In adaptive WM training, the trained tasks are repeated frequently and the cognitive load required for the task is held at the highest possible individual performance level. The magnitude of studies conducted in this area in recent years is enormous and the populations that have been studied are wide, ranging from young children to the elderly and covering a wide variety of neurological or medical conditions, such as ADHD (Gray et al. 2012; Holmes et al.
2010), intellectual disabilities (Van der Molen, Van Luit, Van der Molen, Klugkist, & Jongmans, 2010), and low birth weight (Løhaugen et al., 2011).

According to the meta-analyses conducted in the field, thus far only near transfer has been reliably confirmed (Melby-Lervåg & Hulme, 2013; Shipstead, Hicks, & Engle, 2012a). There are promising studies suggesting far transfer effects, including far transfer in mathematical domains, such as early numeracy in kindergarten children (Kroesbergen, Van ’t Noordende, & Kolkman, 2012, 2014) and word problem solving in elementary school children (Kuhn & Holling, 2014). However, claims of the effectiveness of far transfer effects should not be accepted without caution, due to methodological and theoretical questions raised in previous studies (Shipstead, Hicks, & Engle, 2012a, b).

One widely adopted approach in WM training studies has been to use many different WM tasks in a single training program (Klingberg, 2010). The other approach has been to use one single WM task, for example, an n-back task, which is supposed to train the executive component of WM (Jaeggi, Buschkuehl, Jonides, & Perrig, 2008). The underlying idea behind training with multiple tasks or with one executive task is that it leads to the enhancement of a common, domain-general capacity. However, since thus far only near transfer effects have been reliably proven (Melby-Lervåg & Hulme, 2013; Shipstead, Hicks, & Engle, 2012a), it has been suggested in previous literature that the mechanism that is being trained is not a general, executive capacity but merely a modality specific capacity (Melby-Lervåg & Hulme, 2013), relating either to verbal or nonverbal memory processes.

In this study we investigated the effects of specific WM training on WM, fluid intelligence, and counting skills. We addressed the training strictly to specific WM components and compared their transfer effects. More specifically, we were interested in examining whether the training of one WM component transfers (1) to tasks similar to the trained tasks (domain-specific near transfer, e.g., verbal STM training enhancing performance in verbal STM tasks), or (2) also to other WM tasks (domain-general near transfer, e.g., verbal STM training enhancing performance in verbal WM tasks). In addition, we examined (3) whether the training of any WM components have far-transfer effects on counting skills and fluid intelligence. We used a wide range of criterion measures. For assessing near-transfer, we used two measures to assess each training groups’ domain-specific transfer, and six measures to assess domain-general near transfer. For assessing far transfer we used one measure for assessing fluid intelligence and one for assessing counting skills.
Methods

Participants
The participants in the study were 99 Finnish children (42 girls, 57 boys) from nine kindergartens in the city of Turku. At the time of the pre-training assessments, training, and post-training assessments, all children were attending their last two months of kindergarten. In Finland, kindergarten starts the year the child turns six, preceding compulsory schooling, which starts the year the child turns seven. All research permissions from the city of Turku (Early Childhood and Basic Education Section), kindergarten teachers, and legal guardians of the children were gathered.

There were four children who did not attend the post-training assessments and had missing values also from the pre-training assessments due to scheduling problems in the data collection. These children were removed from the data analysis. We also removed from the data analysis one child who scored 0 on several tasks. The number of participants in the statistical analyses is thus 94.

There were 19 participants who did not attend one or two of the four pre- or post-training assessments sessions due to scheduling problems (illness, holiday trips). The missing data for one or two of the four assessment sessions were more pronounced (19% missing). To overcome the problems of missing data we used multiple imputation, which has been demonstrated to reduce bias and improve efficiency relative to listwise deletion (Rubin, 1996; Graham, 2009). The multivariate imputation by chained equations was carried out in R package MICE (Van Buuren & Groothuis-Oudshoorn, 2011), which imputes data by running a series of regression models. In this approach, each variable with missing data is modeled with logistic or linear regression using the other variables, taking into account the distribution of the variable. The assumptions underlying the imputation are that data are missing at random (Graham, 2009). We used available information on WM and scholastic measures to produce five imputed data sets and pooled the statistical tests accordingly.

Experimental design
The participating children were randomly assigned to six different experimental groups. The groups were visuospatial short-term memory (N = 15), visuospatial working memory (N = 16), verbal short-term memory (N = 17), verbal working memory (N = 14), active controls (N = 15), and passive controls (N = 17). Children in the passive control group took part in the regular kindergarten activities and pre- and post-training assessments.
The pre- and post-training assessments were conducted individually in the quiet room of the kindergarten by four trained research assistants. All the participating children were told that they were going to do different tasks with the research assistant. No mention of the aims of the study was made. The pre-training assessment was conducted in a two-week period before the onset of the training and the post-training assessment in a two-week period after the training. A similar assessment battery was adopted in the pre- and post-training assessments.

The training lasted for five weeks and was instructed by the research assistants. The participating children were told that they were going to play a children’s computer game. Again, the aims of the study were not mentioned. There were two training sessions per week. Each training session contained 24 trials, starting from the beginning of the training task (one item to remember). Since the amount of training was based on the trials, not the time of training, the training times varied with the achieved level and the game played. The training was adaptive.

**Pre- and post-training assessment**

**Working memory**

The WM subcomponents were assessed with the Automated Working Memory Assessment (AWMA; Alloway, 2007). The AWMA is a WM test battery based on Baddeley’s (1986, 2000) WM model, measuring the different WM components with several tasks. Because AWMA is not standardized in Finland or translated into Finnish, only the visuospatial tasks were administered by computerized AWMA. The verbal tasks were adapted to the Finnish language on the basis of the original English version of AWMA.

For each task of AWMA, the test trials were presented as a series of blocks. Each block consisted of six trials. If the participant responded correctly to the first four trials within a block, the task proceeded to the next block and a score of six was given for the completed block. If three or more errors were made within a block, the test was stopped and the score of the total correct responses was given. The sequence started from one word and the length of the sequence increased after every sixth trial. The testing was finished when the participant failed to repeat four out of six trials.

**Visuospatial short-term memory.** In the dot matrix task the participants were presented with a sequence of red dots on a 4 x 4 grid for two seconds each. The child was required to point with a finger to the positions of the dots that had appeared in the order of the appearance. In the block recall task, a sequence of
cubes was highlighted on a screen with nine randomly located cubes. The child’s task was to repeat the sequence in the same order by pointing at them with a finger. The test-retest correlations for dot matrix and block tasks are .85 and .90, respectively (Alloway, 2007).

**Visuospatial working memory.** In the odd-one-out task, the participants were presented with a row of three shapes and instructed to point with a finger to the odd one out and remember its location. At the end of the task, the child was instructed to recall the position of the shape that he or she had identified as being different. In the Mister X task, the child was presented with a picture of two people, each of whom was holding a ball in one of their hands. One person was rotated and the child’s task was to tell if the two people were holding the ball in the same hand and to recall the location of the rotated person’s ball. The test-retest correlations for odd-one-out and Mister X tasks are .88 and .84, respectively (Alloway, 2007).

**Verbal short-term memory.** In the word span forward task, the participants were instructed to recall orally series of word lists in the correct serial order. In the non-word task, the procedure was the same as for the word span forward, except that it consisted of lists of non-words. In the non-word task, credit was given for phoneme substitutions when the experimenter judged that the substitution constituted the child’s habitual articulation pattern for that phoneme. The test-retest correlations for word span forward and non-word span are .88 and .69, respectively (Alloway, 2007).

**Verbal working memory.** In the listening recall task, the participants were presented with a set of spoken sentences and instructed to judge whether a sentence was true or false by answering orally and retaining the final word of the sentence in sequence. In the counting recall task, the children were presented with a visual array of red circles and blue triangles. They were instructed to count the number of circles by pointing at them with a finger, and to recall the tallies of the circles after the array disappeared. In both tasks, the children responded orally. The test-retest correlations for listening recall and counting span tasks are .88 and .83, respectively (Alloway, 2007).

**Counting skills**

Counting skills were assessed with counting tasks from the Diagnostic Tests for Metacognition and Mathematics test battery (Salonen et al., 1994). Eight verbal counting subtasks were adopted from the test. The tasks require counting up to
50, counting the number of objects, counting up from a number, counting up a specific amount from a number, counting backward from a number, and counting backward from one number to another. The test consists of 25 items, which were scored as 1 for a correct choice and 0 for an error. The maximum score is thus 25.

**Fluid intelligence**

Fluid intelligence was assessed with Raven’s Coloured Progressive Matrices (Raven, Court & Raven, 1995). These consist of a series of reasoning tasks, in which the child is required to complete a geometrical figure by choosing a missing piece from six choices. The test consists of 36 items, which are scored as 1 for a correct choice and 0 for an error. The maximum score is thus 36.

Along with these tasks, reading related skills (reading ability, rapid automated naming, phonological awareness, letter naming, and phonological coding) were assessed in the pre-training and post-training assessment batteries. Data from these tasks are not reported here, but the complete data of the pre-assessment will be reported elsewhere.

**WM training**

The four computerized WM training programs were developed for use in this study. Common to the developed training programs was the adaptiveness of the task difficulty, which was matched to the actual performance of each child. The task difficulty was increased or decreased by varying the number of memoranda. The memory-tasks appeared in the blocks of four tasks. If the child made an error in three of the four tasks, she or he dropped to the previous level (one less item to remember). If the child completed three of the tasks, he or she progressed to the next level (one more item to remember). Otherwise the child got the subsequent block of four items at the same level. The smallest possible set size was one. The session started from level one. In each training session, the child had 24 trials.

*Visuospatial STM training.* In the visuospatial STM training, the child was presented with a matrix in which animal figures appeared for two seconds in half of the squares. The child was instructed to recall in which squares of the matrix the animals had appeared and to use a mouse to click on the squares on an empty matrix after their disappearance. The number of squares in the matrix increased by two in every fourth pattern.
Visuospatial WM training. In the visuospatial WM training, the child was presented with three animals located above places to hide (e.g., a tree or a stone). One of the animals was different from the others. The child was instructed to point out which one was different. After that, the animal hid (moved behind the element). The child was expected to retain the position of the hiding places and a new set of animals and hiding places appeared. At the end of the sequence the child was instructed to recall the hiding places in their presentation order by mouse click. In every fourth sequence the number of hiding places increased by one.

Verbal STM training. In the verbal STM training, the child was instructed to learn new words that animals taught. One animal appeared on the screen and the non-word was heard on the headphones. The child was instructed to repeat orally the non-words in the correct order. Stimuli consisted of five letters. The appearance of the non-words was randomly assigned.

Verbal WM training. This training task had verbal storage and processing requirements. In the verbal WM training, the child was instructed to learn new words from fish and to select the fish that had taught the word on the basis of a tip. In this training two fish appeared on the screen and a syllable emerged from the headphones (storage component). After that the child was instructed to point with a finger at the fish that had taught the word (processing component). At the end of a series, the child recalled the syllable/syllables in the order of their presentation orally and the next trial began. In every fourth sequence the number of syllables that the fish taught increased by one.

Active control group activity. The active control group played two computer games (Moomin) that required no memorization. In these games, the child explores the Moomin world visually and aurally. By clicking objects on the computer screen, different things appear and happen. Conversely to memory games, in these games the children were supposed to proceed and find different things, not to stop and memorize what they had already experienced. Participants in the active control group played the game for seven minutes per session.

Results

First, to check potential pre-training differences between the two experimental groups and the control group, one-way ANOVAs comparing the performance in pre-assessment tasks and age were carried out. Second, to test the effectiveness of the interventions, a repeated measures analysis of variance was carried out.
with the group as the factor and the scores on the different tests as the dependent variables. We report the statistical analyses pooled from five imputed datasets with the raw scores. Since the amount of missing information was relatively high, inducing uncertainty as to the precision of the final data, we do not report the exact means and standard deviations in a table. The means and standard errors of the pooled values can be evaluated on the basis of the presented figures. All experimental conditions are reported. The sample size was determined before collecting the data (for recommendations on good research and reporting practices see Simmons, Nelson, & Simonsohn, 2011).

There were no significant differences in performance between groups at the pre-assessment stage on any of the ten measures (all F-values are < 1.01; p-values are non-significant). Next, we investigated the near transfer of WM training by calculating repeated measures ANOVAs between pre- and post-assessment scores for each of the WM tasks. Regarding the near transfer, our results showed no significant Time × Group interaction (all F-values are < 2.52, p-values are non-significant). This means that the change in the performance of any of the WM tasks, either domain-specific or domain-general near-transfer tasks, did not differ statistically in the six groups from pre-assessment to post-assessment and no further analyses were thus conducted. In Figures 1-4 we present the performance of each group in all pre- and post-assessment tasks.

Figure 1. The performance (means) in two visuospatial STM tasks at pre- and post-training assessments in experimental groups. Error bars represent standard errors. The means and standard errors are pooled over five imputed datasets.
Finally, we investigated the far transfer effects of WM training by analyzing the effects of training on performance in the counting task and fluid intelligence task.
For performance in these two tasks, our results showed no significant Time × Group interaction (all F-values are < 2.58; p-values are non-significant; see Figure 5). This means that the change in the performance of these tasks did not differ statistically in the six groups from pre-assessment to post-assessment. All analyses were repeated using complete data (complete case analysis, N = 69) and the results remained similar (all F-values are < 1.61, p-values are non-significant).}

Since our statistical analyses did not show interaction between time and training group in any criterion tasks (i.e., the change in time was not different in the training groups), we do not report the main effects of the experimental group and time or the effect sizes of the training effects.

**Discussion**

The aim of this study was to compare the effects of four different WM training conditions: verbal and visuospatial STM training and verbal and visuospatial WM training and their effects in improving WM and early counting skills in kindergarten children. We used a full WM test battery with eight tasks to assess different subcomponents of WM along with a fluid intelligence test and broad counting test to assess far transfer of training. We also used an active control group along with a passive control group in order to rule out the expectancy and pre-test effects. Our results show that domain-specific WM training did not enhance performance in any of the WM tasks assessing corresponding WM components (domain-specific near-transfer), in WM tasks assessing other WM components (domain-general near transfer), or in tests of fluid intelligence or early counting (far transfer) compared to active and passive control groups. The gain in the experimental training groups did not outperform the re-testing gain. This result indicates that the increase in the performance between pre- and post-assessment conditions was
only associated with the well-documented finding that re-testing leads to higher performance compared to the first testing.

Earlier studies have adopted very intensive and varied training addressing different WM subcomponents in each training session (Klingberg, 2010). It is possible that the modality specific WM training conducted in the kindergarten settings adopted in this study was too weak to obtain training effects in WM and counting. However, it is also possible that the lack of near transfer effects in WM tasks in our study is related to the inability of kindergarten aged children to utilize efficient memory strategies like rehearsal or grouping (Gathercole, Adams, & Hitch, 1994). Recent research has strongly suggested that the enhancement of strategies during training is a strong candidate in explaining near transfer effects found in WM training (Dunning & Holmes, 2013, St. Clair-Thompson, Stevens, Hunt, & Bolder, 2010). Since our participants are younger than participants in many recent studies, they may have had difficulties taking advantage of mnemonic strategies, and there was no solid ground for the enhancement of WM capacity through strategy learning.

Since there were no near transfer effects in our study, far transfer effects are not expected. However, the results are informative in part because of the practice effects. In general, besides the training effects, it is possible that the experimental design with pre-assessment and post-assessment also induces practice effects (Bartels, Wegrzyn, Wield, Ackermann, & Ehrenreich, 2010; Benedict & Zgaljardic, 1998; Müller, Kerns, & Konkin, 2012), which are a consequence of repeated testing. Benedict and Zgaljardic (1998) studied the effect of practice during repeated administrations of verbal and nonverbal memory tests. Participants who were tested with the same items every two weeks improved significantly over four sessions. In our study, the children systematically improved, also in the two control groups, irrespective of the nature of the group. This highlights the importance of controlled experimental studies in confirming the effects of cognitive training.

In this study, we trained different WM components separately for the first time. Our results suggest that six-year-old children are not able to utilize specific computerized WM training in kindergarten settings, and it does not enhance WM and counting skills. The finding of the study is important for the current debate in the field. It is critical to study the boundaries of the amount of training that is required to enhance young children’s WM and counting, the differences in the malleability of specific mechanisms in WM (Logie, 2012), and the real life applications of WM training in kindergarten settings.
Acknowledgements

We thank all the participating children and their parents and kindergarten teachers. We thank Jyrki Messo of Messo Technologies Oy for developing the working memory training software. We also thank Laura Englund, Leena Kittilä, Matleena Korhonen, and Riitta Lahtonen for their help with the data collection, and Jari Lipsanen for assistance with the statistics.

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References


Dansk resumé

Specifik træning af arbejdshukommelse og tællefærdigheder i børnehaveklassen

Børns tælestrategier i førskolealderen hænger tæt sammen med deres arbejdshukommelse og er en god prediktor for deres senere præstationer i skolefaget matematik. Denne artikel bygger på et studie, som sammenlignede fire forskellige slags arbejdshukommelsesøvelser med henblik på at vurdere deres effekt i form af forbedring af arbejdshukommelsen (nær transfer) og tællestrategier (fjern transfer). Hukommelsesøvelserne rettede sig mod træning af verbal og visuospatial korttidshukommelse og mod træning af verbal og visuospatial arbejdshukommelse. Nioghalvfems børn blev tilfældigt fordelt mellem de fire forskellige slags træning, en aktiv kontrolgruppe og en passiv kontrolgruppe. De computerbaserede øvelser foregik to gange ugentligt i fem uger i elevernes vante miljø i børnehaveklassen. Vores resultater viser, at den specifikke arbejdshukommelsesøvelse ikke forbedrede elevernes præstationer i test af arbejdshukommelsen eller i test af tællefærdigheder. Vores studie tyder på, at det for at vurdere en eventuel anvendelighed af forskellige metoder til træning af arbejdshukommelsen er nødvendigt at gennemføre studier, der undersøger, hvor lang tids træning de forskellige metoder i så fald forudsætter, hvis de skal kunne forbedre elevernes arbejdshukommelse og tælleprocesser i børnehaveklassen.

Nøgleord: træning af arbejdshukommelse, tælleprocesser, børn, delelementer i arbejdshukommelse.
Can Children Enhance Their Arithmetic Competence by Playing a Specially Designed Computer Game?

By Ingemar Holgersson, Wolmet Barendregt, Elisabeth Rietz, Torgny Ottosson, & Berner Lindström

Abstract

Fingu is a game and a game platform using virtual manipulatives designed to help children develop competence and fluency with basic number combinations. We present results from an intervention lasting for seven to nine weeks, in which 82 children (5, 6, and 7 years old) were allowed to play the game as part of their ordinary preschool or school activities. The results showed significant positive differences between pre- and post-tests in four arithmetic measures, with moderate to large effect sizes. In contrast, most differences between post-tests and delayed post-tests were non-significant, with low or no effect sizes.

Keywords: number sense, math games, virtual manipulatives, arithmetic competence, perceptual learning.
Introduction

A basic level of mathematical literacy is to master arithmetic concepts and skills and in particular, to understand number concepts and to be able to add and subtract natural numbers. According to Baroody, Bajwa, and Eiland (2009), the learning of arithmetic starts when children are two to four years old with the development of an understanding of the intuitive numbers one, two, and three. An important step in the development of arithmetic competence is to learn to master the basic number combinations, such as $4 + 3 = 7$ or $8 + 2 = 10$, and to be able to use these flexibly in arithmetic problem solving. Mastery with fluency grows out of the development of a “rich network of factual, relational, and strategic knowledge” (Baroody et al., 2009, p. 70). To master the basic part-whole relations between all the numbers in the range of 1 to 10 is therefore a critical foundation for developing an adaptive expertise with numbers. And children who experience difficulties with arithmetic seem to be characterized by having a poorly developed network of such part-whole relations (Gray & Tall, 1994). Instead they become reliant on counting as the only method they can use to arrive at sums and differences. Also, Neuman (1987; 2013) found that resorting to counting as a core method is common among children with mathematical difficulties.

In a project called CoDAC (Conditions and tools for Developing Arithmetic Competence) we investigated the ways in which a certain kind of digital game can support children’s development of such a network of part-whole relations of numbers. Using the latest technology we developed an iPad game called Fingu, aimed at children from four to eight years old. Fingu uses what has been termed virtual manipulatives, which is a design affordance of the iPad technology (and other touch screen devices). Moyer, Bolyard, and Spikell (2002) defined a virtual manipulative as “an interactive, Web-based visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge” (ibid., p. 373). This technology makes it possible to design mathematical learning environments that afford embodied, and not just symbolic, mathematical activities.

Design of Fingu

The basic idea in the game is that the player is exposed to two moving sets of objects and is supposed to tell how many objects there are in total by touching the screen with the same total number of fingers. The basic structure of the task resembles an Initiation–Response–Evaluation (IRE) sequence (Mehan, 1979), with a “problem presentation” in visual mode, an answer given by using fingers, and
feedback about the correctness of the answer by a picture and accompanying sound. However, it is important to appreciate that the task comprises all phases of the sequence. These make up a perception-action cycle, which may include a transformation of the exposed partition of the sum into another partition given by the fingers.

Another key part of the game is that the player is, due to a limited touch input latency (default 0.25 seconds), forced to use a coordinated finger pattern to complete a task. This means that the child/player cannot give the answer by sequentially touching the screen with one finger at a time, but is forced to touch the screen at the same time with all the fingers that constitute the pattern. Thus the player is stimulated to focus on the parts of the presented problem and the total sum, instead of resorting to enumeration techniques for figuring out and presenting the total sum. The degree of freedom in choosing which fingers and which partition to use gives the task the quality of a problem solving activity that allows for exploration. It is primarily not a simple skill-focused activity, directed at rote learning. An opportunity for the children to learn to use their fingers to express a certain partition or a certain sum is therefore an essential part of what Fingu affords. The limited touch input latency also makes it inconvenient to touch the displayed objects one at a time in order to enumerate them, because there will be a high risk of the game “interpreting” such a touch as an answer of one. Counting is thus a less productive strategy.

A progression, or trajectory of learning, is built into the game in a design with seven “levels” of difficulty, with increasing sums and more non-canonical visual patterns of objects.

Although Fingu can be played “as is”, another strength lies in the fact that it is developed as a research platform offering numerous possibilities for adaptation, both by researchers and teachers. Teachers can easily change settings such as the speed with which the patterns move around on the screen, the number of mistakes the player can make before he/she is assessed as having managed a certain level, the exposure time of the patterns, as well as the time for answering. The number of levels and assignments per level can be defined by creating an XML file that can be put into the game. This makes it possible to create special profiles for different groups of children on the basis of the teachers’ requirements. Log files allow teachers to replay and study how children use the game. These ways of adjusting the game make it possible to vary the game to better suit individuals with specific needs.
The broader theoretical underpinnings in this study are given by Gibsonian ecological psychology (Gibson, 1986), and in particular the theory of perceptual learning (cf. Gibson & Pick, 2000). Acknowledging the contributions of both James and Eleanor Gibson, Gibsonian theory is intrinsically relational and non-dualistic. It is also non-representational, in the sense that it rejects the idea that perception and cognition are about constructing representations of the world “outside” the individual.

The Gibsons argue against what they term enrichment theories, where perception or “sensory reception is enriched and supplemented by the addition of something” (Gibson & Pick, 2000, p. 7). The alternative that is proposed is that “perception begins as unrefined, vague impressions and is progressively differentiated into more specific percepts” (ibid., p. 7). In development, through perceptual learning, the individual becomes more and more apt to learn the specific affordances of the environment.

This view also builds on the idea that there is a concrete physical environment that individuals act in, perception being the selection and “picking up” of information in the environment. For the present purposes, it is enough to acknowledge that this kind of reasoning is grounded in a “realist” ontology.

The idea that differentiation is basic to learning and development is consequential. In the present context it means that the development of flexible and adaptive competence in dealing with “part-whole” relations in the range of 1 to 10 is a matter of successive refinement of the understanding of numbers, from a more undifferentiated “whole” (e.g., the number 7) to grasping a differentiated “web” of relations between “parts” that can make up the whole (i.e., 7 can be decomposed into 6\|1; 5\|2; 4\|3; 3\|3; 1 etc.). The whole can take on different forms, for example, “number words” of a more symbolic nature; sets or constellations of concrete objects, visually presented “patterns” (of objects), or even procedures (of which the counting sequence is an example).

Gibsonian theory also emphasizes that perception is not static but a dynamic process. It is an intrinsic part of how we engage with an environment, for example, by walking around on a beach or playing a computer game. The construct “perception-action cycles” captures this dynamic, emphasizing that perception is not a prerequisite for action. Rather, action is foundational for perception and perception is foundational for action, making up a perceptual system.

Another important aspect of perceptual (and cognitive) systems is that they are typically not uni-modal. They are multi-modal, building on the use of several sensory modalities. This is important in the present context, where children are
exploring a game environment that explicitly draws on both visual and kinaesthetic modality. In this view, embodiment builds on multi-modal agency.

The concept of affordance, which takes on a number of different definitions in contemporary social and behavioural sciences, is pivotal to Gibsonian theory and was developed to account for the relational nature between the individual and the environment. It “refers to the ‘fit’ between an individual’s capabilities and the scaffolds/support and opportunities that make a certain activity possible” (cf. Gibson & Pick, 2000, p. 15). An affordance can be thought of as an offer of meaning in a given situation, or put in more general terms, an offer of action.

Perceptual learning is about discovering meaning; it is:

... the means of discovering distinctive features and invariant properties of things and events (E.J. Gibson, 1969). Learning to distinguish faces from one another or to distinguish letters of the alphabet are such cases. Discovering a repeated theme in a symphony and the variations on it is another. Discovering distinctiveness and invariance is another kind of meaning, also a product of perceptual learning. (Gibson, 2000, p. 295)

The idea of discovering and picking up information as invariant features of the environment is central in James Gibson’s seminal work (Gibson, 1986) and is in our understanding very similar to what variation theory proposes (Marton & Pang, 2006). This should be emphasized in this study since precursors of the Fingu game were designed using ideas that were precursors to the variation theory (cf. Lindström, Marton, Emanuelsson, Lindahl, & Packendorff, 2011).

Perceptual learning builds on two complementary processes or systems: exploratory activity and performatory activity. To cite Eleanor Gibson:

Exploratory activity … is itself an event, a perception-action sequence that has consequences. It brings about new information of two kinds: information about changes in the world that the action produces and information about what the active perceiver is doing. (Gibson, 2000, p. 296)

This kind of activity is flexible and geared to maintain an adaptive relation with the environment, learning what options for action there are. But learning is also geared towards economy and efficiency. In the beginning, an activity can be explorative and different actions are tested. But when an activity develops a limited set of actions are chosen, which are more effective and better suited to fulfilling the goals of a task. This is what the theory of perceptual learning conceptualizes as performatory activity:
Activity that starts as exploratory can become performatory as an affordance is discovered. This shift is marked by making contact with the environment and ensuing control of it. (ibid., p. 297)

Building on the theory of perceptual learning, Kellman et al. (2008) have formulated design principles for interventions aimed at developing expertise with some key areas in mathematics learning. These principles include: many and varied short tasks, where the child has the opportunity to develop rapid selection of task-relevant information, and the pick-up of higher-order relations and invariances in different modalities such as visual, auditory, and kinaesthetic.

Aims

One of the aims of the CoDAC project is to investigate how children come to understand part-whole-relations in the number range of 1 to 10, and more generally, arithmetic competence. Another aim, which is specifically targeted in this article, is if playing a tailored computer game, Fingu, can support this development. Of special interest is to investigate whether children’s game playing enhances general arithmetic competence.

Methodology

The study reported here has used a pre-, post-, and delayed post-test design and involved 82 children (30 five-year-olds, 25 six-year-olds, and 27 seven-year-olds), who for seven to nine weeks were allowed to play Fingu as one part of their practice in preschool or school. Before this experimental period, we measured their arithmetic abilities by using a set of arithmetic test instruments (pre-test), of which we report the results of four (see below). These data were collected by interviewing the children individually and video-recording parts of the interviews. We repeated the measurements when the intervention period was over (post-test), as well as around eight weeks later (delayed test). Children were not allowed to play Fingu during the period between the post-test and the delayed test.

In the measuring procedure we used four measures of arithmetic ability: the Test of Early Mathematics Ability, version 3 (TEMA-3), a test of part-whole knowledge (PWK), a problem solving test (PS), and a pattern recognition test (PR).

TEMA-3 is an American test developed by Ginsburg and Baroody (2003) that we translated into Swedish for this study. It measures children’s general arithmetic abilities and includes sub-tests on counting, computations, number comparisons, and problem solving. It is normed for American children, but here we only use raw scores as a measure of the children’s general arithmetic ability.
The PWK test consists of six tasks that indirectly test the children’s knowledge of different part-whole number relations. Most of the tasks focusing on answering with finger patterns to represent different numbers as well as testing the children’s knowledge of the combinations of 5 and 10 were adopted from Clarke, Clarke, and Cheeseman (2006), but one task called ‘the guessing game’ was adopted from Neuman (1987).

The PS test consists of eight verbally presented story problems that invite the children to present their solutions. All the problems are of an addition or subtraction type and have sums ≤ 10. They are a representative selection of problems that have been used for another investigation of a computer game using a specially designed keyboard that can be said to be a forerunner to Fingu (see Lindström et al., 2011), and it is thus possible to compare two related, but different, games.

Lastly, the PR test is a specially constructed test consisting of 20 tasks presented using the Fingu platform to investigate the children’s knowledge of a representative selection of ten of the single set number patterns used in the game. Each pattern is only exposed for 0.5 seconds, and the answer is given verbally and recorded by the interviewer.

Other sets of data, not reported in this article, that we have gathered consist of log files that record data, such as the exposed task, the child’s answer, the answering time, and the location of the fingers used to answer the task. These log files can also be used to replay a session in Fingu, where the finger locations appear as red dots, making the finger use visible. To complement these data, we have also video recorded three sessions with each participant, documenting how they use their fingers while playing the game.

Results

Schools and preschools were instructed to let the children play at least three times a week, but there is great variation in the total number of times and the length of time the children have played (Table 1). The teacher’s role was only to offer the opportunity to play Fingu, but it was up to the individual child to decide if and to what extent they wanted to play. As can be seen from Table 1, comparing the medians, the five-year-old children have on average solved 25% more tasks than the six-year-old children, and around 30% more tasks than the seven-year-old children. The dispersion, measured as the interquartile range, in the five-year-olds’ distribution was however much larger than for the six-year-olds and the seven-year-olds, and there was one exceptional outlier with 4572 tasks in total (i.e. ≈ 183 tasks per day or ≈ 13 rounds per day using Fingu).
Table 1:

Number of tasks played by the children during the intervention period, for different ages and in total.

<table>
<thead>
<tr>
<th>Age</th>
<th>N</th>
<th>Median</th>
<th>Q₁</th>
<th>Q₃</th>
<th>IQR</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 years</td>
<td>30</td>
<td>1112</td>
<td>651</td>
<td>1739</td>
<td>1088</td>
<td>186</td>
<td>4572</td>
</tr>
<tr>
<td>6 years</td>
<td>25</td>
<td>889</td>
<td>575</td>
<td>1127</td>
<td>552</td>
<td>279</td>
<td>1833</td>
</tr>
<tr>
<td>7 years</td>
<td>24</td>
<td>857</td>
<td>668</td>
<td>1358</td>
<td>690</td>
<td>320</td>
<td>2445</td>
</tr>
<tr>
<td>All</td>
<td>79</td>
<td>918</td>
<td>646</td>
<td>1366</td>
<td>720</td>
<td>186</td>
<td>4572</td>
</tr>
</tbody>
</table>

Note. Q₁ = Lower quartile, Q₃ = Upper quartile, and IQR = Interquartile range. The total number of 79 is due to three missing cases of log files among the seven-year-olds.

To answer the question of whether playing Fingu gives children any benefits in ordinary school mathematics, we performed analyses using paired-sample t-tests to determine whether the mean values of the different instruments are significantly different, and Cohen’s d to determine the effect sizes of these differences. The results are presented in Tables 2 and 3.

Table 2:

Results of a paired-sample t-test of the mean values of the different pre- and post-tests, complemented by effect sizes.

<table>
<thead>
<tr>
<th>Test</th>
<th>N</th>
<th>Pre-test</th>
<th>Post-test</th>
<th>Difference</th>
<th>Effect size</th>
<th>t</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tema3</td>
<td>81</td>
<td>28.15</td>
<td>31.40</td>
<td>3.25</td>
<td>0.34</td>
<td>7.54</td>
<td>.000***</td>
</tr>
<tr>
<td>PWK</td>
<td>82</td>
<td>15.88</td>
<td>18.65</td>
<td>2.77</td>
<td>0.42</td>
<td>6.33</td>
<td>.000***</td>
</tr>
<tr>
<td>PS</td>
<td>74</td>
<td>4.08</td>
<td>5.28</td>
<td>1.20</td>
<td>0.46</td>
<td>5.17</td>
<td>.000***</td>
</tr>
<tr>
<td>PR</td>
<td>75</td>
<td>11.64</td>
<td>14.44</td>
<td>2.80</td>
<td>0.79</td>
<td>8.54</td>
<td>.000***</td>
</tr>
</tbody>
</table>

In Table 2 the values of the pre-tests are compared with the values of the post-tests. The number of individuals varies due to some missing values. What we found is that all the instruments have significant differences with \( p < 0.001 \). Effect sizes were evaluated according to Cohen’s original suggestion of 0.2 as small, 0.5 as moderate, and 0.8 as large. This means that there was a small effect of the intervention in TEMA-3, a moderate effect in PWK and PS, and a large effect in PR.

In Table 3 the values of the post-tests are compared with the delayed tests performed around eight weeks after the intervention. Here we find a highly significant (\( p < 0.001 \)) difference in the mean values in TEMA-3 combined with a small effect size. For the PWK instrument there is a less significant difference (\( p < .05 \)) in the mean values, but also a small effect size. For the PS instrument there is
no effect and no significant difference in mean values, and for the PR instrument there is no significant difference in mean values, but a small effect size.

Table 3:

Results of a paired-samples t-test of the mean values of the different post- and delayed-post-tests complemented by effect sizes.

<table>
<thead>
<tr>
<th>Test</th>
<th>N</th>
<th>Post-test</th>
<th>Delayed test</th>
<th>Difference</th>
<th>Effect size</th>
<th>t</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>TEMA3</td>
<td>82</td>
<td>31.43</td>
<td>33.16</td>
<td>1.73</td>
<td>0.18</td>
<td>4.64</td>
<td>.000***</td>
</tr>
<tr>
<td>PWK</td>
<td>80</td>
<td>18.65</td>
<td>19.68</td>
<td>1.03</td>
<td>0.16</td>
<td>2.38</td>
<td>.020*</td>
</tr>
<tr>
<td>PS</td>
<td>81</td>
<td>5.12</td>
<td>5.09</td>
<td>-0.04</td>
<td>-0.01</td>
<td>-0.22</td>
<td>.829</td>
</tr>
<tr>
<td>PR</td>
<td>78</td>
<td>14.46</td>
<td>13.97</td>
<td>-0.49</td>
<td>-0.15</td>
<td>-1.76</td>
<td>.082</td>
</tr>
</tbody>
</table>

Our interpretation of these results is that TEMA-3 and PWK both measure some ongoing development of arithmetical knowledge that however, was increased during the intervention period. For PS there was a moderate effect during the intervention period that did not continue after the intervention. For PR there was a large effect during the intervention period that turned into a small but non-significant effect in the period after the intervention.

If we perform the same kind of analyses for the different age groups we find some interesting differences. They show that there were significant differences with \( p < 0.01 \) in the mean values, with modest to large effect sizes between pre- and post-tests for the different age groups of five-year-olds, six-year-olds, and seven-year-olds, in that order:

- TEMA-3 \( (d = 0.38; 0.32; 0.60) \),
- PWK \( (d = 0.43; 0.34; 0.75) \),
- PS \( (d = 0.56; 0.48; 0.54) \), and
- PR \( (d = 0.84; 0.88; 1.02) \).

Studying the differences in mean values between the post-test and the delayed post-test for the different age groups resulted in the following effect sizes:

- TEMA-3 \( (d = 0.20; 0.29; 0.23) \),
- PWK \( (d = 0.36; 0.19; 0.02) \),
- PS \( (d = -0.01; -0.09; 0.02) \), and
- PR \( (d = -0.03; -0.39; -0.11) \).

All effects are low or negative. The only significant differences in mean values appeared for all the age groups in TEMA-3 \( (p \approx 0.01) \), but only for the five-year-olds in PWK \( (p < .05) \).
Comparing the results from the pre-tests, post-tests and delayed post-tests we found that all the age groups gained from the intervention, but the seven-year-olds gained more than the other groups on TEMA-3, PWK, and PR. The reason that we did not see this phenomenon in the PS test can be attributed to a ceiling effect in this test. In the post-intervention period we found comparable and small effect sizes for all the age groups in TEMA-3. The only difference between age groups was that the five-year-olds continue to gain on the PWK test. Why this was the case is not evident to us.

We have also performed some analyses where the children have been grouped into low, medium, and high competency on the basis of their pre-test-results in TEMA-3. An analysis showed that the effect sizes were similar for all these groups, that is, all performance levels gained from playing *Fingu*.

**Discussion**

In this article we have presented the empirical background and the theoretical rationale for an iPad game called *Fingu*. This game has been specially designed to address a specific problem in the learning of arithmetic: the necessity for a child to build up a network of part-whole relations of the numbers in the range of 1 to 10. This is necessary in order to develop fluency with basic number relations and an adaptive expertise with computations and other uses of numbers. This build-up of a network of number relations can be seen as a watershed that distinguishes low-performing children in mathematics from the others. The theoretical foundations that underlie our design principles support such an endeavour.

We have also presented preliminary results from an intervention using the game in a naturalistic setting, in both preschool and school. We found that the mean values of performance in four different arithmetic tests increased significantly during the intervention period, with effect sizes that are small, moderate, or large. After the intervention period, however, changes in the comparable test results show only small effect sizes, and for only two of the tests, TEMA-3 and the PWK test, was the difference in the mean values statistically significant.

Looking at the results for the different age groups we found that all the groups perform significantly better on the post-test than on the pre-test. The seven-year-olds, however, gained more than the others. This may mirror the fact that in Sweden children enter school when they are seven years old and are for the first time exposed to a systematic mathematics curriculum.

So far we have only detected correlations between taking part in the intervention and the mean gains in mathematical performance as measured by the four tests; gains that are not fully matched in the post-intervention period. However,
when looking into correlations between immediate parameters characterizing how different children succeed with *Fingu*, such as the number of tasks performed or levels managed, we do not find any significant correlations. Therefore, we cannot claim that a simple causal relation exists between playing time and the gains in test performances. We also found that the individual differences in how children used the game were large. In order to understand in more detail how playing *Fingu* promotes arithmetic competence, more detailed analyses of individual cases are needed.

Note
1 The CoDAC project was funded by the Swedish Research Council (project number 721-2009-5996).

References


Svensk sammanfattning

Kan barn förbättra sin aritmetiska kompetens genom att spela ett särskilt utformat datorspel?

Fingu är ett spel och en spelplattform som används så kallade virtual manipulatives, vilka har designats för att hjälpa barn att utveckla kompetens och flyt i att hantera grundläggande talombinationer. I artikeln presenterar vi resultat från en interventionsstudie, där 82 barn (5, 6 och 7 år gamla), under sju till nio veckor fick spela Fingu som en del av den ordinarie verksamheten i förskolan, förskoleklassen eller år 1 i skolan. Resultaten visade signifikanta, positiva skillnader mellan för- och eftertest på fyra olika sorters aritmetiska test med moderata till stora effektstorlekar. Som jämförelse var skillnaderna mellan eftertest gjorda direkt efter interventionen och eftertest gjorda efter ytterligare en period på cirka åtta veckor, då barnen inte hade möjlighet att spela Fingu, ej signifikanta skillnader eller signifikanta skillnader men med betydligt mindre effektstorlekar. Slutsatsen blir att barnen under interventionsperioden förbättrade sina aritmetiska kompetenser på ett sätt som inte motsvarades av de förbättringar av dessa kompetenser som skedde under den period då de inte hade möjlighet att spela Fingu.

Nyckelord: taluppfattning, matematikspel, virtuella redskap, aritmetisk kompetens, perceptuellt lärande.
Number Sense as the Bridge to Number Understanding

By Lisser Rye Ejersbo

Abstract

In this article, I compare number sense, which is understood as an innate capacity to know about the magnitude and relations of numbers, with number understanding, which is understood as the ability to manipulate the symbolic arithmetic developed in a culture. My research focuses on the question of how number sense and number understanding can be used in a synergetic process in learning mathematics. Different research results have shown that consistent training in number sense (K-10) influences mathematical competencies in a positive manner, and that young children aged 5-6 years are able to solve symbolic problems that involve the approximate addition and subtraction of large two-digit numbers. Our experiment was conducted in a kindergarten class (age 5-6 years) and in a Grade 2 class (age 7-8 years) with students who had difficulties in number reading and symbolic arithmetic.

Keywords: number sense, number understanding, approximation, estimation, kindergarten class and Grade 2.

Introduction

In 2014, the ‘Brain Prize’ from the Danish Lundbeck Foundation was awarded to Stanislav Dehaene. His research has been important in revealing how
children understand and learn mathematics, particularly the fact that children are born with a sense of numbers (for more information in Danish, see Ejersbo & Steffensen, 2013). In his studies, Dehaene (1997) investigates how numbers are stored in the memory of children who are only a few months old. The results of these studies demonstrate that, from birth, there are different areas in the brain available to recognize the size or quantity of entities. Dahaene calls this inherited ability number sense. This type of number sense can be used to differentiate between magnitudes without instruction. However, the concept ‘number sense’ is understood differently across the scientific field with a biological/cognitive view being adopted in some research groups and an educational/pedagogical view in other research groups, of course with some overlap; furthermore, authors from the different groups do not always explain that the expression ‘number sense’ is used in different ways.

Dehaene represents the biological/cognitive view and subscribes to the idea that human infants have a rudimentary number sense: an intuitive understanding of numbers, their magnitude, and relations (Dehaene, 1997). This sense is as basic as our perception of colour, and it is wired into the brain from birth (Piazza, 2011).

An educational/pedagogical description of number sense is different:

[It] reflects a set of understandings and skills that enable a person to look at a problem holistically before confronting details, look for relationships among numbers and operations and ... consider the context in which a question is posed; choose or invent a method that takes advantage of his or her own understanding of the relationships between numbers or between numbers and operations and ... seek the most efficient representation for the given task; use benchmarks to judge number magnitude; and recognize unreasonable results for calculations in the normal process of reflecting on answers. (Andrews & Sayers, 2015, p. 258, quotation from B. Reys)

The understandings and skills described in the educational/pedagogical view of number sense are developed through teaching; I term it number understanding in this article.

My research focuses on how to bridge the gap between the knowledge of number sense gained from the biological/cognitive area and the educational/pedagogical view of number sense or, as I term it, number understanding. This bridging is embedded in what is called pedagogical neuroscience (Sunde & Ejersbo, 2014; Steffensen & Ejersbo, 2014). To investigate this, I have designed several experiments based on previous research in this area. The experiment described in this article is inspired by Gilmore, McCarthy, and Spelke (2007),
who showed how children aged 5-6 years were able to solve different symbolic arithmetic tasks with large two-digit numbers. I wanted to repeat the experiment to investigate whether the same results would be obtained for Danish children. Danish two-digit numbers are pronounced inversely to how they are written: the ones are said before the tens (for instance, 21 is pronounced “one and twenty”). Danish number words are very old and mirror a number concept that is primitive in relation to mathematical thinking (Talord, 2014). Repeating a research project has some benefits. For example, it is possible to compare the results, and the experiment can be developed to meet the requirements of the new context.

Number sense

As a theoretical background, I use Dehaene’s description of number sense. The brain area for rudimentary number sense has been found in different mammals, which is shown through brain scanning under certain conditions. The area or module is located in the Intra Parietal Sulcus (IPS) in the Parietal lobe. The module called ‘number sense’ enables us to sense exactly the size of a number, to compare numbers, and to determine the larger amount between two sets. It is an evolutionarily derived sense, and is a part of our natural intuition. The innate conception of number sense is closely related to the approximation number system (ANS). Piazza (2011) explains ANS as a basic feature of the environment in which animals are able to make a spontaneous extraction of the approximate number of objects in sets. In an evolutionary perspective, this sense is beneficial, making it possible to recognize the number of in-group versus out-group members. A similar spontaneous detection of the approximate number in sets has also been reported in humans. According to Piazza, the most important definition of ANS is that it represents numbers in an approximate and compressed fashion, making it possible to distinguish between two sets if they differ from each other by a given ratio; this is also called the smallest noticeable difference.

The challenge is to find the means by which to use this knowledge in teaching elementary mathematics. Lourenco, Bonny, Fernandez, and Rao (2012) showed that higher mathematical skills are related to ANS competencies, and longitudinal studies have shown that consistent training in ANS provides suitable results for number understanding, as well as for higher-level mathematics (Halberda, Mazzocco, & Feigenson, 2008).

A meta-analysis indicates that it is possible to distinguish between the active locations in the brain using large or small numbers. The hypothesis also includes the notion that the type of brain activation varies with the number of operands involved; for instance, number detection and comparison activate the brain
differently than more complex calculations (Dehaene, Molko, Cohen, & Wilson, 2004).

Children’s play or school work with ANS appears to be crucial in the understanding of abstract numerical concepts that are uniquely human. Piazza (2011) describes the relation between ANS and number understanding as follows:

> One crucial step toward the construction of a representation of exact numbers is achieved when children understand the counting principles, thanks to which they can perform exact quantification on sets of any cardinality, thus overcoming the low resolution of ANS acuity. (ibid. p. 275)

Working in schools with ANS is a good start in terms of stimulating number understanding. Knowing about ANS and how to develop a trajectory from ANS to the stages of symbolic number understanding is important information for teachers in mathematics.

Dehaene (1997) also argues that the Arabic numerals are encoded in the mind as a mental number line oriented from left to right. The mental number line was originally considered a metaphor; however, lately, a different view suggests that:

> [T]he mental number line is not simply a metaphor but … the mental representation of numerical magnitude is considered to be homeomorphic to the representation of psychical space. (Fias, van Dijck, & Gevers, 2011, p. 135)

The image of such mental number line places is similar to an automatic, unconscious process, which means that it is also a part of the intuitive system.

However, learning mathematics is a cultural phenomenon. Therefore, the important issue is to determine how to build on this innate number sense to develop a broader understanding of numbers, which can lead to higher mathematical skills.

**Number understanding**

As noted earlier, the expression ‘number sense’ has different meanings, depending on whether it is being used by biological/cognitive researchers or by educational/pedagogical researchers. Number understanding can be trained in many different ways. In Denmark, teachers are free to choose whatever methods they prefer to use. The chosen methods are often connected to the individual teacher’s knowledge and beliefs. The teachers will attempt to build on the knowledge they believe the children already have when they begin school. However, the question is, how much does the teacher actually know about the innate number sense and the manner in which children develop their number understanding?
Research has shown that at 3-5 years of age, children more or less understand the five counting principles, at least until the set of 10, even when they err in their counting. The five counting principles are (Gelman & Gallistel, 1978; Siegler, 2003, p. 290):

1. The one-one principle: Assign one and only one number word to each object.
2. The stable order principle: Always assign the numbers in the same order.
3. The cardinal principle: The last count indicates the number of objects in the set.
4. The order irrelevance principle: The order in which objects are counted is irrelevant.
5. The abstraction principle: The other principles apply to any set of objects.

Children typically learn the names of numbers as a long list of words and demonstrate knowledge of the stable order principle by nearly always saying the number words in a constant order and saying the last number with emphasis (ibid.). The names are developed as sounds connected to the number of objects in the sets.

The developmental shift to understanding the number name as a cardinal value requires a qualitative shift in children’s conception of numbers. The cardinal principle requires an understanding of the logics behind counting (Goswami, 2008) and the ability to judge the size of a set. The principle relies on a representation of quantitative information in which the coding of smaller quantities is different from the coding of larger quantities (ibid.). Understanding numbers is often connected to teaching, as described for the pedagogical number sense. Andrew & Sayers (2015) describe what they call Foundational Number Sense (FONS). FONS contains eight key components:

- Number recognition
- Systematic counting
- Awareness of the relationship between number and quantity
- Quantity discrimination
- An understanding of different representations of number
- Estimation
- Simple arithmetic competence
- Awareness of number patterns

This description of FONS includes both the biological and pedagogical understanding of number sense. Estimation is connected to ANS and thus, is also a part of the biological number sense. Number sense also includes the ability to
differentiate between sets of objects; this is what I use in the process of teaching students to gain a better number understanding. However, most children who are not able to count are able to point at the group with the most non-symbolic things, for instance, dots. The work with numbers as symbols moves from IPS to the frontal lobe and later to the angular gyrus (Klingberg, 2011). One explanation for the activation of different modules is that the processes change from being mainly analytical to mainly automatic, where the answers come quickly from the memory. In children up to the age of 12, we observe that arithmetic thinking occurs in the frontal lobe and involves the working memory (ibid.).

Number sense and number understanding differ because number sense and approximation skills are inherited, whereas developing the components in FONS involves a cultural learning process. This finding means that understanding numbers is an analytic process, as long as it is not turned through practice into a more automatic process. In particular, knowing the number names in the addition or multiplication table of ten is often learned by rote in the Danish school culture. The hypothesis is that number sense is a type of instinct, which can be a bridge to number understanding. Being a novice in number understanding means that the knowledge is not yet integrated as an automatic skill, although this can be achieved through teaching. When the components in FONS become automatic, the different components will appear quickly and intuitively in the mind. Moving from being a novice to an expert is a process that involves the brain modules, in which we can observe and measure activity when people solve tasks.

An experiment involving ANS

Gilmore, McCarthy, and Spelke (2007) showed experimentally how it was possible for young children at the age of 5-6 years to solve certain symbolic arithmetic tasks with large two-digit numbers embedded in a text. In the experiment, the children were told a story that contained approximate symbolic arithmetic problems. The children, who had not yet been taught symbolic arithmetic, were given problems such as: “If you had twenty-four stickers and I gave you twenty-seven more, would you have more or less than thirty-five stickers?”

Children performed well without resorting to guessing or comparison strategies that could serve as alternatives to arithmetic. The conclusion of the experiment was that children’s approximate arithmetic performance evidently does not depend on knowledge of exact numbers. Furthermore, Gilmore, McCarthy, and Spelke, who are based in the UK and the USA, note that arithmetic instruction in most primary school curricula focuses primarily or exclusively on exact operations involving small numbers. The researchers wrote:
Most children take years to master the set of exact, single-digit addition and multiplication facts, and mastery of these facts is vulnerable to fatigue or interference even in adults. … Our findings suggest that the difficulty of learning and performing arithmetic stems from the demands of constructing and operating on representations of exact number: representations beyond the limits of precision of non-symbolic number representations. (Gilmore et al., 2007, p. 590)

Inspired by this experiment and the results, I designed a small-scale experiment. My intention was to investigate whether Danish children/students reacted in the same way as the children in the experiment of Gilmore et al. (2007).

Our experiment

The first experiment was a pilot study. The methods used in this experiment were planning, designing tasks, observation, and an interview with the teacher. This experiment took place in a group of four children with special needs in Grade 2. The students were offered two extra math lessons per week over a period of two months. The four students had difficulties with counting and understanding number concepts. Two students had a mother tongue other than Danish; however, the language used was Danish. The students were asked mental arithmetic questions, for instance, which number is larger, embedded in a story, as in the Gilmore et al. experiment. In the stories, one, two, or three children had candy and obtained more or gave some away. The task was to decide which set of candy was larger/smaller than the other sets. Only two-digit numbers were used, from the lowest numbers up to 99. Before the experiment, I asked the teacher if she thought that the students would be able to answer correctly; she doubted that they would be able to. She was surprised that all four students were able to solve the problems easily. The students were able to give the correct results very quickly with both large numbers and smaller numbers. The students smiled and wanted to have more of these tasks, which they managed very well. It appeared to be a victory for these students that they were very good at these estimation tasks. During the two-month period with the extra lessons, the students’ understanding of the concepts seemed to improve. This pilot study provided the frame for the design of the next investigation.

The methods in the second experiment consisted of planning, designing, observing, and interviewing the teacher and each of the 16 students in a kindergarten class. The future Grade 1 mathematics teacher conducted the interviews in accordance with my instructions. Ten of the children had a mother tongue other than Danish.
In Danish kindergarten classrooms, the children normally count a lot and perform tasks with small numbers. In our experiment, the students received tasks with large two-digit numbers up to 99. These were more or less the same tasks we used in the pilot study, in which the students were asked to estimate who had the most candy. At the end of the experiment, we added tasks with smaller numbers from 1 to 20.

The tasks were provided orally to one child at a time. The tasks were variations of the following types in addition, subtraction, and comparison, taken from the Gilmore et al. experiment (ibid. p.589):

1. Anna has 21 candies; she gets 30 more. Bo has 34 candies. Who has more?
2. Anna has 64 candies; she gives 13 of them away. Bo has 34 candies. Who has more?
3. Anna has 51 candies. Sarah has 64 candies. Bo has 34 candies. Who has more candies, Anna or Bo?

In the experiment, we varied the tasks so that the numbers to be compared were very close to each other to investigate the smallest noticeable difference. One example of a task was: “Anna has 23 candies; she gets 10 more. Bo has 39 candies. Who has more?”

All of the students answered quickly and correctly most of the time; even when the difference was as small as in the above example.

When asking the students how they arrived at the results, the typical answer was that it just came to them and that the task was easy. Some of the students could not answer the question regarding how they obtained the results and answered: “I don’t know”.

In the addition tasks, we changed the size of the numbers as follows: “Anna has 7 candies, and she gets 5 more. Bo has 14 candies. Who has more?”

When changing tasks to contain small numbers, we noticed that the students went through another process, in which they calculated these tasks using another strategy than simply arriving at the result intuitively. With this task, most of the students took longer to solve each task.

Answering the question regarding how they obtained the results, the students could explain how they calculated their answers. One of the students explained how she used her fingers, and she said it is difficult when the result was higher than 10. Another counted from 7 to 12 and could then observe that Bo had more than Anna, whereas other students simply stated that they could observe it.

The investigation of Danish children appeared to produce the same results as those published by Gilmore et al. In the next stage, in addition to our own results (which confirmed those of Gilmore et al.), we also referred to the work
of Halberda, Mazocco, and Feigenson (2008). This study recommends that math teaching include estimation tasks throughout the school system. For our investigation, we designed different tasks and instructions for this type of teaching.

Other Estimation Tasks

To explore how to train estimation, the teacher developed different courses to let the students estimate on a daily basis. In the beginning, primarily numbers of objects were used; however, later, weight and length were also added.

Below are two examples, tasks 1 and 2 from Grade 1, which focus on estimating the number of objects:

**Task 1:**

![Figure 1](image1)

This picture is shown on the whiteboard for a few seconds before the teacher says: “Those of you who think that there are more white cubes than grey cubes, raise your hand”.

The same procedure is conducted with the opposite colour. The next question is why the students chose their answers. Once the picture has been shown for a while, the teacher asks some of the students to count the white cubes and others to count the grey cubes. The strategies used for counting the cubes are very different, which could prove interesting to investigate in the future. The teacher’s task during the process is to encourage the students to count groups of cubes and ask them where they will begin counting and where they will end.

The second task builds on task 1. The pattern recognition in task 1 can be used in task 2; however, task 2 can also be solved on its own.
The teacher repeats the procedure from task 1. In their explanations, some of the students refer to task 1, whereas others view it as a new task.

The learning goal established for the lesson was to train the students’ estimation skills for numbers up to 30. In addition, the goal was to support students’ further learning of counting items, developing a grouping for the exact results of amounts. Finally, through the mathematical communication in the class and the special questions from the teacher, the goal was to develop the students’ understanding of numbers (FONS) using both their number sense and number understanding in combination.

We also developed other types of tasks; for instance, one task invited students to determine the weight of different items. In one example, the teacher uses a bag of potatoes and assigns a number to each potato. The students’ tasks are to decide which potatoes weigh more than others and then to place them in a number line from the lightest to the heaviest.

Developing tasks for this type of investigation continues to occur in a longitudinal study, and the students are being followed with a focus on how they develop their number skills and understanding, according to the description of FONS. In this study, different designs are attempted; using estimation to develop number understanding is one of these designs.

Discussion

Our experiment indicates that Danish children with no formal arithmetic instruction were able to perform symbolic addition, subtraction, and comparison with
large two-digit numbers when they were asked to estimate the results in several tasks. We also observed that the students in kindergarten took a longer time to reach an answer when working with smaller numbers; we interpret this to indicate an activation of a different brain module than the IPS, where number sense is located. Calculation for children in this age group remains unautomated, which is why the activation occurs in the frontal lobe with the use of their working memory (Klingberg, 2011). The speed of solving the tasks also provides insight into the working memory load. It appears that the load on working memory is much lower when students work intuitively using their number sense than when they work analytically with calculation and counting. The experiment’s objective is to train the students’ number sense as well as number understanding, on the basis that this will improve the integration of the two processes.

**Implications for school practice**

In primary schools, students usually work with counting and exact calculations much more than with estimation. To my knowledge, working with exact calculations is often more difficult for many of the students in this age group. A more analytic process is required to understand numbers sufficiently well to obtain an exact result, and this increases the load on the working memory. From the results of Gilmore et al. and our experiment, it appears that a different process is taking place when students answer intuitively rather than using their analytical skills. Working with their number sense, the children were very fast and provided correct results. In addition, it was clear that providing the correct answer made the students feel happy and confident, particularly because they found solving the tasks easy.

My research focuses on bridging the gap between research on number sense from a biological/cognitive perspective on the one hand, and from an educational/pedagogical perspective on the other. Building on the experiment conducted by Gilmore et al., we designed new tasks to further explore the bridges between biology, psychology/cognition, and pedagogy: the so-called pedagogical neuroscience paradigm. The objective was to investigate whether, and how, teaching in school can include knowledge as number sense or the mental number line to support students’ development of analytical skills for understanding numbers; this is also something included in FONS (Andrew & Sayers, 2015).

For this inclusion to occur, curricula that allow children to use their number sense by comparing numbers and by using their mental number line need to be developed. Such a curriculum may involve, for instance, children being asked to estimate before calculating or being provided with special training in which the
goal is to extend their number sense in order to enhance their number understanding.

The results of our experiment may not surprise researchers; however, they do surprise teachers. Danish mathematics teachers are not trained in number sense or taught that children are able to work with approximation and estimation. Therefore, teachers do not expect children to be able to estimate large two-digit numbers. Teachers need to know about these topics to develop ideas that could help their students develop number understanding more easily by using their number sense. This approach should be considered during lessons and in curriculum planning.

The results presented in this article are the first to emerge from this longitudinal study, which will include further tasks designed to bridge the gap between number sense and number understanding. These results will be reported at a later date.

Conclusion

The results of the experiment indicate that it may be beneficial for students if we develop a curriculum that brings number sense into the classroom and makes it a part of the daily work with numbers. A crucial question is how we prepare such a curriculum and use it practically in teaching. It is not enough to consider number sense in isolation; it must be connected to number understanding and to the other components of FONS. Finally, this area of study is not only important for primary school; number sense and intuitive thinking need to be integrated into mathematics teaching at all levels and for all children.

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References


Dansk resumé

Talsans som bro til talfornemmelse og talforsøg

Kan mindre børn på 5-6 år regne med store tal op til 99 og vurdere, hvornår resultatet af en beregning er større eller mindre end et andet tal? Denne artikel fokuserer hovedsageligt på en undersøgelse af dette spørgsmål, samt på hvordan undervisning i matematik i de mindre klasser kan se ud, hvis man inddrager denne viden i sin planlægning og udførelse af undervisningen.

Vi bliver født med en talsans, som giver os evnen til at skelne mellem forskellige antal (Dehaene, 1997). Udvikles denne talsans bevidst, kan den støtte udviklingen af børns talfornemmelse og talforsøg, forstået som evnen til at manipulere med tal og vide, hvornår hvilke strategier er hensigtsmæssige at anvende for at løse et talproblem. Gilmore, McCarthy og Spelke (2007) har gennemført forskellige eksperimenter, som viste, at 5-6-årige børn er i stand til at svare korrekt på opgaver med større tal op til 99, både opgaver med addition, subtraktion og sammenligning, når opgaven var at estimere, hvilket tal der var det største.

Inspireret af Gilmore, McCarthy og Spelkes resultater gennemførte jeg sammen med forskellige lærere eksperimenter, der skulle vise, om danske børnehaveklasseelever også var i stand til at afgøre, hvilket tal der var størst eller mindst i opgaver med tal op til 99. Opgaverne, vi brugte i undersøgelsen, var af typen: ”Anna har 21 stykker slik. Bo har 35. Anna får 30 stykker mere. Hvem har flest?”

Det viste sig, at eleverne i børnehaveklassen havde let ved at svare korrekt på sådanne spørgsmål. I forskningsprojektet arbejdede vi derfor videre med at udvikle og afprøve forskellige typer af opgaver til eleverne i de mindre klasser med det formål at udnytte deres talsans til at udvikle deres talfornemmelse og talforsøg.

Nøgleord: Talsans, talfornemmelse, talforsøg, estimering, 0. og 2. klasse.
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